WEAK LANDAU–GINZBURG MODELS FOR FANO THREEFOLDS AND THEIR PROPERTIES

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1. Reminder

Fano variety in the following means smooth variety X with ample $-K_X$ and $\mathrm{Pic}\,(X)=\mathbb{Z}$.

Given Fano variety X one may associate *Gromov–Witten invariants* with it — numerical invariants counting genus 0 curves lying on X. Given these invariants one may construct *regularized quantum differential equation* (or *regularized quantum $\mathcal{D}$-module*) Q_X.

Given a pencil $Y\to\mathbb{C}$ one may associate *a Picard–Fuchs differential equation PF_Y* with solutions of type $\int_{\Delta_t}\omega_t$, where Δ_t and ω_t are fiberwise cycle and form of middle dimension.

Conjecture 1 (Mirror Symmetry of Hodge structures variations)**.** *For any Fano variety X there exists pencil $Y\to\mathbb{C}$ such that $Q_X\cong PF_Y$.*

In the following we consider threefold case.

Given Fano threefold X let

$$a_{ij}=\langle(-K_X)^i,(-K_X)^{3-j},-K_X\rangle_{j-i+1},\quad 0\le i\le j\le 3,\ j>0,$$

be a Gromov–Witten invariant whose meaning is an expected number of rational curves of anticanonical degree $j-i+1$ that intersect general representatives of cohomological classes dual to $(-K_X)^i,(-K_X)^{3-j},-K_X$. These numbers are known for all 17 Fano threefolds.

Consider a ring $\mathcal{D}=\mathbb{C}[t,\frac{\partial}{\partial t}]$ and a differential operator $D=t\frac{\partial}{\partial t}\in\mathcal{D}$. Then regularized quantum differential operator for X is given by the following equality.

$$\begin{aligned}
&L_X=D^3-t\,(2\,D+1)\,(\lambda D^2+(a_{11}+\lambda)D^2+\lambda D+(a_{11}+\lambda)D+\lambda)\\
&+t^2\,(D+1)\,((a_{11}+\lambda)^2D^2+\lambda^2D^2+4\,(a_{11}+\lambda)\lambda D^2-a_{12}D^2-2\,a_{01}D^2\\
&+8\,(a_{11}+\lambda)\lambda D-2\,a_{12}D+2\,\lambda^2D-4\,a_{01}D+2\,(a_{11}+\lambda)^2D+6\,(a_{11}+\lambda)\lambda\\
&+\lambda^2-4\,a_{01})-t^3\,(2\,D+3)\,(D+2)\,(D+1)\,(\lambda^2(a_{11}+\lambda)+(a_{11}+\lambda)^2\lambda-a_{12}\lambda+a_{02}\\
&-(a_{11}+\lambda)a_{01}-a_{01}\lambda)+t^4\,(D+3)\,(D+2)\,(D+1)\,(-\lambda^2a_{12}+2\,a_{02}\lambda+\lambda^2(a_{11}+\lambda)^2\\
&-a_{03}+{a_{01}}^2-2\,a_{01}(a_{11}+\lambda)\lambda)\,,
\end{aligned}$$

defined up to a shift $\lambda\in\mathbb{C}$. It has a unique analytic solution $I^X_{H^0}$ of type $1+a_1t+a_2t^2+\ldots$ called *the fundamental term of the regularized I-series* of X. Moreover,

$$Q_X\ \Leftrightarrow\ I^X_{H^0}\ \Leftrightarrow\ \text{Gromov–Witthen theory of } X.$$

Assume now that $Y=(\mathbb{C}^*)^3$. Then $f\colon Y\to\mathbb{C}$ may be represented by Laurent polynomial.

Theorem 1. *Let ϕ_i be a constant term of f^i. Then there exist particular Δ_t and ω_t such that*

$$\int_{\Delta_t}\omega_t=1+\phi_1t+\phi_2t^2+\ldots$$

We call this series *the constant terms series* and denote by $\Phi_f(t)$.

Thus Conjecture 1 reduces to the following one.

Conjecture 2. *For any Fano threefold X there exists Laurent polynomial f such that $I^X_{H^0}=\Phi_f$.*

Such f is called *a very weak Landau–Ginzburg model* for X.

The same may be done for any dimension.

The assumption on Y looks restrictive, but it turns out that it is not restrictive at all.

Conjecture 3. *Any Fano variety has a very weak Landau–Ginzburg model.*

Later we discuss some conditions on very weak Landau–Ginzburg models; these conditions strengthen this conjecture.

All known Landau–Ginzburg models may be represented by Laurent polynomials.

- Smooth Fano complete intersections in Grassmannians have very weak Landau–Ginzburg models.
- Smooth Fano complete intersections of Cartier divisors in weighted projective spaces have very weak Landau–Ginzburg models.
- Smooth Fano threefolds have very weak Landau–Ginzburg models.

2. General picture

Theorem 2. *Each Fano threefold has a particular very weak Landau–Ginzburg model.*

We call these models *standard*.

There are a lot of very weak Landau–Ginzburg models for given Fano threefold even if we identify those that differ by toric coordinate changes. Most of them are "wrong" from the point of view of another Mirror Symmetry conjectures (such as HMS). So the natural question is: how to fix the "correct" ones?

The first step is to involve some basic expectations of HMS. It assumes that the dual Landau–Ginzburg model is a pencil of relative compact Calabi–Yau varieties. The general element of the pencil is smooth; vanishing cycles to singular elements form a category equivalent to $\mathcal{D}^b(X)$. We expect that such Landau–Ginzburg models are compactifications of our non-compact ones.

Definition 1. A very weak Landau–Ginzburg model f is called *weak* if for almost all $\lambda\in\mathbb{C}$ the hypersurface $Y_\lambda=\{f=\lambda\}$ is birationally isomorphic to Calabi–Yau variety.

Example 1. Consider a standard model for $\mathbb{P}^3$. The general element of the pencil given by it is

$$\left\{x+y+z+\frac{1}{xyz}=\lambda\right\}\in(\mathbb{C}^*)^3\times\mathbb{C}.$$

Compactifying it using embedding $(\mathbb{C}^*)^3\hookrightarrow\mathbb{P}^3$ we get a quartic

$$\left\{(x+y+z)xyz+t^4=\lambda xyzt\right\}$$

in $\mathbb{P}^3$ with Du Val singularities. Thus the general element is birational to K3 surface.

The second step is based on Batyrev's approach of small toric degenerations of Fano varieties. Let Fano variety X have small toric degeneration — a degeneration to terminal

N.	Index	Degree	Description	Weak LG model
1	1	2	Sextic double solid V_2' (a double cover of $\mathbb{P}^3$ ramified over smooth sextic).	$\frac{(x+y+z+1)^6}{xyz}$
2	1	4	The general element of the family is quartic.	$\frac{(x+y+z+1)^4}{xyz}$
3	1	6	A complete intersection of quadric and cubic.	$\frac{(x+1)^2(y+z+1)^3}{xyz}$
4	1	8	A complete intersection of three quadrics.	$\frac{(x+1)^2(y+1)^2(z+1)^2}{xyz}$
5	1	10	The general element is V_{10}, a section of $G(2,5)$ by 2 hyperplanes in Plücker embedding and quadric.	$\frac{(1+x+y+z+xy+xz+yz)^2}{xyz}$
6	1	12	The variety V_{12}.	$\frac{(x+z+1)(x+y+z+1)(z+1)(y+z)}{xyz}$
7	1	14	The variety V_{14}, a section of $G(2,6)$ by 5 hyperplanes in Plücker embedding.	$\frac{(x+y+z+1)^2}{x} + \frac{(x+y+z+1)(y+z+1)(z+1)^2}{xyz}$
8	1	16	The variety V_{16}.	$\frac{(x+y+z+1)(x+1)(y+1)(z+1)}{xyz}$
9	1	18	The variety V_{18}.	$\frac{(x+y+z)(x+xz+xy+xyz+z+y+yz)}{xyz}$
10	1	22	The variety V_{22}.	$\frac{xy}{z}+\frac{y}{z}+\frac{x}{z}+x+y+\frac{1}{z}+4 +\frac{1}{x}+\frac{1}{y}+z+\frac{1}{xy}+\frac{z}{x}+\frac{z}{y}+\frac{z}{xy}$
11	2	$8\cdot 1$	Double Veronese cone V_1 (a double cover of the cone over the Veronese surface branched in a smooth cubic).	$\frac{(x+y+1)^6}{xy^2z}+z$
12	2	$8\cdot 2$	Quartic double solid V_2 (a double cover of $\mathbb{P}^3$ ramified over smooth quartic).	$\frac{(x+y+1)^4}{xyz}+z$
13	2	$8\cdot 3$	A smooth cubic.	$\frac{(x+y+1)^3}{xyz}+z$
14	2	$8\cdot 4$	A smooth intersection of two quadrics.	$\frac{(x+1)^2(y+1)^2}{xyz}+z$
15	2	$8\cdot 5$	The variety V_5, a section of $G(2,5)$ by 3 hyperplanes in Plücker embedding.	$x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+xyz$
16	3	$27\cdot 2$	A smooth quadric.	$\frac{(x+1)^2}{xyz}+y+z$
17	4	64	$\mathbb{P}^3$.	$x+y+z+\frac{1}{xyz}$

TABLE 1. Weak Landau–Ginzburg models for Fano threefolds.

Gorenstein toric variety. Let $v_1, \ldots, v_k \in \mathbb{Z}^n$ be integral generators of its fan. Then the suggested weak Landau–Ginzburg model is

$$x^{v_1} + \ldots + x^{v_k},$$

where multidegrees are taken in a standard way.

We expect that this effect holds in a more general way. That is we expect that if f is a "correct" weak Landau–Ginzburg model then the initial Fano variety X degenerates to a toric variety such that the linear span Δ of integral generators of rays of its fan is the Newton polytope of f. Hilbert polynomials of X and its toric degenerations should equal. In the threefold case this means that the volume of the polytope dual to Δ equals $(-K_X)^3/3!$.

Appropriate toric degenerations of Fano threefolds are studied not well. As a particular result in this way there is the following theorem.

Theorem 3 (S. Galkin). *There are exactly 5 of 17 smooth Fano threefolds admitting degenerations to terminal Gorenstein toric varieties: $\mathbb{P}^3$, smooth quadric, smooth complete intersection of 2 quadrics, V_5, V_{22}.*

Despite on this one may assume that there is an appropriate toric degenerations to use generalized Batyrev's approach for finding of weak Landau–Ginzburg models.

Example 2. The projective space $\mathbb{P}^3$ is toric itself. The Newton polytope of its standard model is the linear span of vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,-1,-1)$. These vectors are primitive vectors of the fan of $\mathbb{P}^3$. The volume of the dual polytope is $64/3!$.

Definition 2. A weak Landau–Ginzburg model f for Fano variety X is called *semiweak* if the volume of the polytope dual to the Newton polytope of f equals $(-K_X)^3/3!$.

Theorem 4. *The standard Landau–Ginzburg models for Fano threefolds are semiweak.*

We expect that these conditions guarantee that weak Landau–Ginzburg model is "correct". More particular we have conjectural "optimistic picture".

Optimistic picture. Flat toric degenerations of Fano threefolds are in one-to-one correspondence with "correct" weak Landau–Ginzburg models (up to toric changes of coordinates). The minimal compactifications of these models (they differ by flops) are Landau–Ginzburg models in the sense of HMS.

3. Properties and examples

The following property is based on L. Katzarkov's recent idea to relate the Hodge type of Fano variety to the structure of the cental fiber of dual Landau–Ginzburg model and the sheaf of vanishing cycles to this fiber.

Theorem 5. *All minimal compactifications of given standard Landau–Ginzburg model differ by flops.*

Theorem 6. *Let k_X be the number of components of fiber over 0 of minimal compactification of standard Landau–Ginzburg model for Fano threefold X. Then $k_X = h^{12}(X) + 1$.*

Question 1. *Is it true that this equality holds for all semiweak Landau–Ginzburg models for Fano threefolds?*

It is natural for many reasons to restrict us to semiweak Landau–Ginzburg models with Newton polytopes corresponding to canonical toric varieties (in the other words, varieties such that linear spans of integral generators of rays of their fans contain one strictly internal integral point). For threefolds of large degree restrict us also to Gorenstein Newton polytopes[1].

Example 3. There are 5 reflexive polytopes of volume $\frac{64}{3!}$. Three of them give the following weak Landau–Ginzburg models for $\mathbb{P}^3$.

$$x+y+z+\frac{1}{xyz},$$

$$\frac{(x+1)^2}{xyz}+\frac{y}{z}+z,$$

$$x+\frac{y}{x}+\frac{z}{x}+\frac{1}{xy}+\frac{1}{xz}.$$

All fibers of their minimal compactifications are irreducible and smooth except for 4 fibers over roots of degree 4 from 256. All of them are K3 surfaces with one ordinary double point.

Example 4. There are 5 reflexive polytopes of volume $\frac{54}{3!}$. Four of them give the following weak Landau–Ginzburg models for smooth quadric.

$$\frac{(x+1)^2}{xyz}+y+z,$$

$$x+y+z+\frac{1}{xy}+\frac{1}{xz},$$

$$\frac{(x+y)^2}{x}+\frac{1}{yz}+\frac{z}{x}+\frac{y}{xz},$$

$$\frac{(x+1)^3}{xyz}+\frac{y}{z}+\frac{2}{z}+\frac{2x}{z}+\frac{z^2}{y}.$$

All fibers of their minimal compactifications are irreducible and smooth except for 3 fibers over roots of degree 3 from 108 and a fiber over 0. All of them are K3 surfaces with one ordinary double point.

The similar picture holds for Fano threefolds with non-trivial intermediate Jacobians.

Example 5. Consider a complete intersection of 2 quadrics in $\mathbb{P}^5$. Consider two following weak Landau–Ginzburg models for it.

$$\frac{(x+1)^2(y+1)^2}{xyz}+z,$$

$$x+y+z+\frac{1}{xyz}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+xyz.$$

Their minimal compactifications (it is more convenient to change variables $a=\frac{1}{x}$ in the second case for compactifying) one may get pencils of $K3$ surfaces. Both of them have 2 fibers with ordinary double points (over ± 8) and the central fiber consisting of 3 components. Singularities of the central fiber (the intersection of its components) are 3 rational curves. All these curves intersect in 2 points.

[1]Unfortunately we can't always do this. For instance, there are no such polytopes of volume $2/3! = 1/3$ that should correspond to sextic double solid.

Remark 1. Singularities of semiweak Landau–Ginzburg models we get agree with expectations of Homological Mirror Symmetry. That is, a derived category of $\mathbb{P}^3$ is generated by $\mathcal{O}(1)$, $\mathcal{O}(2)$, $\mathcal{O}(3)$, and $\mathcal{O}(4)$, a derived category of quadric is generated by $\mathcal{B}$, $\mathcal{O}(1)$, $\mathcal{O}(2)$, $\mathcal{O}(3)$, where $\mathcal{B}$ is a category generated by 1 element, and a derived category of complete intersection of 2 quadrics is generated by $\mathcal{D}^b(C)$, $\mathcal{O}(1)$, $\mathcal{O}(2)$, where C is a curve of genus 2.

Remark 2. Coordinates of singular fibers of weak Landau–Ginzburg model are determined by its Picard–Fuchs equation. However the number of components of fiber over 0 does not. The example is the following. Let X be a complete intersection of 2 quadrics in $\mathbb{P}^5$. Consider the following weak (but not semiweak and corresponding to toric variety with Picard number 3!) Landau–Ginzburg model for X:

$$\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right).$$

The number of components over 0 of its minimal compactification is 30, but $h^{12}(X) = 2$.

Summarizing, the conjectural picture is the following. For any Fano variety we may associate a set of polytopes with one strictly internal integral point and given volume of dual polytope. For any such polytope there is a weak Landau–Ginzburg model. Given these model we can reconstruct Gromov–Witten invariants of initial Fano variety, its index, degree, characteristic numbers of a general hyperplane section, and the dimension of intermediate Jacobian.

So the last point needed to fix to put everything of quantitative level is the following: how to get a weak Landau–Ginzburg model if we know its Newton polytope? We suggest the following beta-version of the answer.

(1) Put 1's on vertices of the polytope.
(2) Put $\binom{k}{m}$ at m-th integral point form the edge of length $k+1$.
(3) Consider facets of the polytope. If a facet is equilateral triangle (w. r. to the integral length) with length of edge $k+1$, put the corresponding coefficient of expansion of $(x+y+z)^k$ on each integral point of the facet we put nothing yet. If the facet is not equilateral triangle put 0's to all points we put nothing yet.

Question 2. *May this recipe be generalized in a natural way to the higher dimensions?*

The talk is based on papers "Weak Landau–Ginzburg models for smooth Fano threefolds" and "On Mirror Symmetry and intermediate Jacobians" (Przyjalkowski). References for the talk are therein.

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