On the instability of equilibrium in a potential field

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1. We consider the canonical system of equations

$$\dot{x}=\frac{\partial H}{\partial y},\quad \dot{y}=-\frac{\partial H}{\partial x};\quad (x,y)\in\mathbf{R}^{2n} \tag{1}$$

with infinitely differentiable Hamiltonian $H = K(x, y)+\Pi(x)$, where $K = \langle A(x)y, y\rangle/2$ is a positive definite quadratic form and $\langle,\rangle$ is the usual scalar product in $\mathbf{R}^n$. Let $x = 0$ be a critical point of the function $\Pi(x)$, $\Pi(0) = 0$. The system (1) then has the obvious solution $(x, y) = (0, 0)$, which is called a state of equilibrium (the point $x = 0$ is an equilibrium position). If the potential energy Π has a strict local minimum at an equilibrium position, then the equilibrium is stable (Lagrange's theorem).

2. By means of a suitable canonical linear transformation, we can achieve that in the new variables (which are renamed x and y) the kinetic energy is given by $K = \langle y, y\rangle/2+\langle B(x)y, y\rangle/2$, $B(0) = 0$.

Lemma 1. (see [1], [2]). *Suppose that $x = 0$ is not a local minimum of $\Pi(x)$ and that in the domain $U_\varepsilon^- = \{x:\ \Pi(x) < 0,\ \ |x| < \varepsilon\}$ there is a smooth vector field $v(x)$ such that:*

1) $\langle v, \Pi'\rangle \leqslant 0$ *in* U_ε^-;
2) $\langle v'\xi, \xi\rangle \geqslant c\langle \xi, \xi\rangle$ *for all* $\xi \in \mathbf{R}^n$ *and* $x \in U_\varepsilon^-(c > 0)$;
3) $|v(x)| \geqslant \alpha(|x|)$, $\alpha \to 0$ *as* $|x| \to 0$.

Then there is an $\varepsilon_0 > 0$ such that if $(x(t), y(t))$ is a solution of (1) with negative total energy $H(x(t), y(t))$ and $x(0) \in U_{\varepsilon_0}^-$, then $|x(t)| > \varepsilon_0$ for some $t > 0$.

To prove this, we consider a new vector field $w(x) = v(x)-\sigma\Pi'$. If σ is small and positive, it is clear that $\langle w'\xi, \xi\rangle \geqslant \sigma_1\langle \xi, \xi\rangle$ $(\sigma_1 > 0)$. By 1), $\langle w, \Pi'\rangle \leqslant -\sigma\langle\Pi', \Pi'\rangle$. Since $\Pi'(0) = 0$, the vector field w satisfies 3). We put $f(t) = \langle w(x(t)), y(t)\rangle$. Then

$$\dot{f}=\langle w'y,\ y\rangle-\langle w,\ \Pi'\rangle+\langle w'By,\ y\rangle-\langle w,\ \langle B'y,\ y\rangle\rangle/2. \tag{2}$$

Since $B(0) = 0$ and $|w(x)| \to 0$, as $|x| \to 0$, it follows from (2) that $\dot{f} \geqslant \sigma_1|y|^2/2+\sigma|\Pi'|^2 \geqslant \theta > 0$ for $|x| \leqslant \varepsilon_0$ (ε_0 small). If $|x(t)| \leqslant \varepsilon_0$ for all $t > 0$, then clearly $f(t)$ is bounded. But this contradicts the fact that $\dot{f} \geqslant \theta > 0$.

Remark. Let $x(t)$, $y(t)$ be a solution of (1) with zero total energy. If $x = 0$ is an isolated equilibrium position, then (under the hypotheses of Lemma 1) every motion $x(t)$ either leaves a certain domain $|x| \leqslant \varepsilon_0$ in a finite time interval, or tends to zero, as $t \to \infty$ (see [1]).

3. The following result is proved using Lemma 1.

Theorem. *Let $x = 0$ be a critical point of the analytic function $\Pi(x)$ that is not a local minimum. Then the equilibrium position $(x, y) = (0, 0)$ is unstable if one of the following conditions holds:*

A) $\Pi(x)$ *is quasi-homogeneous,*
B) $\Pi(x)$ *is semi-quasi-homogeneous.*

4. A polynomial $\Pi(x_1, \dots, x_n)$ is called quasi-homogeneous of degree $s \in \mathbf{N}$ with exponents $\alpha_1, \dots, \alpha_n \in \mathbf{N}$ if $\Pi(\lambda^{\alpha_1}x_1 \dots \lambda^{\alpha_n}x_n)= \lambda^s\Pi(x_1 \dots x_n)$ for all $\lambda \in \mathbf{R}$. We put $v(x) = \Lambda x$, where $\Lambda = \operatorname{diag}(\alpha_1, \dots, \alpha_n)$. The field $v(x)$ satisfies the conditions 2) and 3) of the instability lemma. Condition 1) follows from "Euler's formula": $\langle\Pi', \Lambda x\rangle = s\Pi$.

5. An analytic function $\Pi(x)$ is said to be semi-quasi-homogeneous if $\Pi = \Pi_0+\Pi_1$, where Π_0 is non-degenerate and quasi-homogeneous of degree s (with $x = 0$ as an isolated critical point), and the degree of monomial in the Maclaurin series of Π_1, regarded as a quasi-homogeneous function with the same exponents, is strictly greater than s. Not every quasi-homogeneous polynomial is semi-quasi-homogeneous.

We introduce in $\mathbf{R}^n$ the "quasi-homogeneous" norm $|x|_* = |x_1|^{1/\alpha_1}+\dots+|x_n|^{1/\alpha_n}$. Evidently, $\Pi_1 = o(|x|_*^s)$. Since $\Pi'(0) = 0$, it follows that $s > \alpha_i$ $(1 \leqslant i \leqslant n)$. For any $m \in \mathbf{N}$ and small values of $|x|_*$ we have

$$c\,|x|_*^{2m}\leqslant\sum_{i=1}^{n}\left(\frac{\partial\Pi}{\partial x_i}\right)^{2m/(s-\alpha_i)}\leqslant C\,|x|_*^{2m} \tag{3}$$

where c and C are positive constants. Let $v(x) = \Lambda x - \sigma V(x)/|x|^{2m-s}$, where

$$V(x)=\left\{\ldots,\ \left(\frac{\partial\Pi}{\partial x_i}\right)^{\frac{2m}{s-\alpha_i}-1},\ \ldots\right\},\qquad m=(s-\alpha_1)\ldots(s-\alpha_n).$$

For small values of σ and ε the conditions 2) and 3) of Lemma 1 hold. Since

$$\langle v,\ \Pi'\rangle = s\Pi_0+o(|x|_*^s)-\sigma\sum\left(\frac{\partial\Pi}{\partial x_i}\right)^{2m/(s-\alpha_i)}\Big/|x|_*^{2m-s},$$

it follows from (3) that when $|x|_*$ is sufficiently small, $\langle v, \Pi'\rangle \leqslant s\Pi - \sigma_2|x|_*^s < 0$, where $\sigma_2 = c\sigma/2$.

6. Using the above theorem we can deduce some known results on the instability of equilibrium positions in potential fields (see [2], Theorem 2 of [3], and [4], for example).

References

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