Calculus of variations in the large and classical mechanics

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CONTENTS

Introduction

One of the basic objects of classical mechanics is the Lagrangian system; a pair (M, L), where M is a smooth manifold (the configuration space of the system), and L is a smooth function on the tangent bundle TM (the Lagrange function or Lagrangian). One can also consider the more general case when L depends explicitly on the time t. Motions of a Lagrangian system are (by definition) smooth paths $x : [t_1, t_2] \to M$ that are critical points of the functional (the Hamiltonian action)

$$F = \int_{t_1}^{t_2} L\,dt$$

in the class of paths with fixed end-points (Hamilton's principle). In local coordinates $x = (x^1, \ldots, x^n)$ motions can be defined as solutions of Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \qquad (1 \leqslant i \leqslant n).$$

The left-hand sides of these equations transform covariantly under a change of local coordinates. Consequently, the set of numbers $(L'_{\dot{x}^i})^\cdot - L'_{x^i}$ can be regarded as a covector, the so-called Lagrangian derivative of L, which is usually denoted by $[L]$.

In the simplest and most common case of "natural" mechanical systems the Lagrangian is given by the function $\langle \dot{x}, \dot{x}\rangle/2 - V(x)$, where $\langle\, ,\rangle$ is a

Riemannian metric on M (twice the kinetic energy) and $V: M \to \mathbf{R}$ the potential of a force field. According to Maupertuis' famous principle of least action the trajectories of the motions with total energy $\langle \dot{x}, \dot{x} \rangle/2 + V(x) = h$ are geodesic lines in the Jacobi metric $(h - V)\langle \dot{x}, \dot{x} \rangle$. Because of the energy integral the motion takes place in the domain where $h - V$ is non-negative. If $\sup_M(V) < h$, then the description of the motions of a natural system reduces to a problem of Riemannian geometry. From the point of view of oscillation theory the most interesting case is when[(1)] $V = h$ at some points of M. At these points the Jacobi metric is degenerate. The first non-trivial results on the behaviour of the trajectories of a mechanical system with a degenerate Jacobi metric were obtained by Seifert [1]. In contrast to another paper by Seifert (on the same topic) on closed trajectories on the three-dimensional sphere [2], this paper unfortunately remained little known for a long time. The active study of domains of possible motions $\{x \in M: V(x) \leqslant h\}$ began after the publication of [3] and [4]. The main attention was concentrated on the problems of the existence of closed trajectories with end-points on the boundary $\{x \in M: h - V = 0\}$. These results are given in §§1-2.

If we add to the Lagrangian $L = \langle \dot{x}, \dot{x} \rangle/2 - V(x)$ a term $\omega(\dot{x})$, where ω is some 1-form on M, then we obtain the next more complicated class of mechanical systems. These are studied in §3. The Lagrange equation contains an additional term, the covector $[\omega(\dot{x})] = \Omega(\dot{x}, \cdot)$, where the 2-form Ω is the exterior differential of the 1-form ω. The presence of this term has no influence on the conservation of energy. If the equations of motion are written in the form $[L(\dot{x})](\cdot) = \Omega(\cdot, \dot{x})$, then Ω can be treated as an additional force acting on the natural mechanical system. We may also consider the more general case when Ω is closed but not necessarily exact. In mechanics, Ω is usually called the form of gyroscopic forces. Their character can be very diverse. Gyroscopic forces appear on transition to rotating systems of coordinates, on lowering the Rouse number of degrees of freedom of a system with symmetries (see, for example, [5]), and also in describing the motion of a charged particle in a magnetic field. Because of the diverse reasons for the presence of gyroscopic forces, the question of periodic motions is very complicated. Substantial progress in this question was made recently by Novikov [6]-[8], who constructed an analogue of Morse theory for many-valued functionals. Novikov's theory refers precisely to the case when Ω is not exact. Questions of the existence of periodic trajectories are also considered in §3 from the point of view of the theory of dynamical systems.

In §§1-3 we use a variational principle for the stationarity of the abbreviated Maupertuis action, which is a consequence of Hamilton's principle and is valid only for autonomous systems. In §4 we establish by means of Hamilton's principle the existence of asymptotic motions.

[(1)]See, for example, [35] or [36].

The results of this section are applied to investigate the stability of periodic or almost periodic regimes of oscillations. For example, we establish rigorous sufficient conditions for the instability of the vertical equilibrium of a pendulum on a vibrating base. Results of this sort are usually obtained by an approximate analysis (for example, by the method of averaging).

It is not our aim to give an exhaustive survey of the applications of the calculus of variations in the large to classical mechanics. We have restricted ourselves to the Lagrangian aspects of mechanics, leaving on one side the variational principle of Hamiltonian systems

$$\delta \int_{t_1}^{t_2} (y\cdot\dot{x} - H)\, dt = 0$$

In this variational problem due to Hamilton, modified by Poincaré ([9], Ch. 29), the symplectic coordinates x, y ("momentum coordinates") are regarded as independent variables. The "action" functional, defined on curves in the phase space, is unbounded below (and above), therefore, the gradient descent method of Morse theory in the problem of periodic trajectories is not effective in this situation. Here other methods are used, of which [10] and [11] give an idea. We mention also Percival's "non-traditional" principle [12], based on work of Mather [13]. This principle was intended for the search for invariant tori of Hamiltonian systems that are close to integrable. Nor do we touch on the applications of variational methods to the theory of complete integrability of the equations of motion of mechanical systems. These questions are discussed in [14] (see also [15]).

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§1. The geometry of domains of possible motions

1.1. The principle of stationarity of the abbreviated action.
Let (M, L) be a Lagrangian system with the Lagrangian $L = L_2 + L_1 + L_0$, where L_s is a smooth function on TM that is homogeneous of degree s with respect to velocity. We assume that L_2 is positive definite so that L_2, the kinetic energy, determines a Riemannian metric on M. The function $L_0 : M \to \mathbf{R}$ can be identified with the force function $U : M \to \mathbf{R}$ ($U = -V$).

The Lagrange equation $[L] = 0$ has the energy integral $H = L_2 - L_0$. For a fixed value $H = h$ the motion can take place only in the domain $B_h = \{x \in M\colon V \leqslant h\}$, the so-called domain of possible motions. For $h > \bar{h} = \sup_M V$ the set B_h is the whole configuration space M. If $h < \bar{h}$, then $\partial B_h \neq \varnothing$. In the typical case when h is a regular value of $H : TM \to \mathbf{R}$ the domain B_h is a smooth manifold with smooth boundary $\partial B_h = \Sigma_h$ whose dimension is 1 less than that of M.

For simplicity let $h = 0$ (if $h \neq 0$, then we can replace L by $L + h$). We assume that $B \setminus \Sigma \neq \varnothing$ (here $B = B_0$, $\Sigma = \Sigma_0$).

Definition. The functional

$$F^* = \int_{t_1}^{t_2} (2\sqrt{L_0 L_2} + L_1)\, dt = F - \int_{t_1}^{t_2} (\sqrt{L_2} - \sqrt{L_0})^2\, dt,$$

defined on smooth curves $x : [t_1, t_2] \to B$, is called the *abbreviated action* or *Maupertuis action.*

The integrand in F^* is a homogeneous function of velocity of degree 1. Consequently, the value of F^* does not depend on the parametrization of the path of integration.

***Theorem* 1.** *A smooth path x: $[t_1, t_2] \to B \setminus \Sigma$ such that $H(\dot{x}(t)) = 0$ for all $t_1 \leqslant t \leqslant t_2$ is a solution of $[L] = 0$ if and only if it is a critical point*[1] *of F^*.*

Proof. Let $[L]_{x(t)} = 0$ and $L_2(\dot{x}(t)) \equiv L_0(x(t))$. Then

$$(1) \qquad \delta F^* = \delta F - 2 \int_{t_1}^{t_2} (\sqrt{L_2} - \sqrt{L_0})\, \delta (\sqrt{L_2} - \sqrt{L_0})\, dt = 0.$$

Conversely, let x: $[s_1, s_2] \to B \setminus \Sigma$ be a stationary point of F^*. We put

$$t = \int_{s_0}^{s} \sqrt{L_2}/\sqrt{L_0}\, d\tau.$$

Then, obviously, a smooth path $x(s(t))$: $[t_1, t_2] \to B \setminus \Sigma$ satisfies $L_2 = L_0$. If $\delta F^* = 0$, then it follows from (1) that $\delta F = 0$. This completes the proof.

We define a Riemannian metric $\langle , \rangle$ in the interior of B by putting

$$\langle \dot{x}, \dot{x} \rangle = 4 L_0(x) L_2(\dot{x}), \qquad \dot{x} \in TB.$$

This is called the Jacobi metric. For natural systems when the form of gyroscopic forces is $L_1 \equiv 0$, Theorem 1 means that in $B \setminus \Sigma$ the motions with zero total energy are the geodesic lines in the Jacobi metric.

When $h > \bar{h}$, then B coincides with M and $(B, \langle , \rangle)$ is an ordinary Riemannian manifold. Otherwise the boundary Σ of B is non-empty and the Jacobi metric has a singularity: the lengths of curves in Σ are zero.

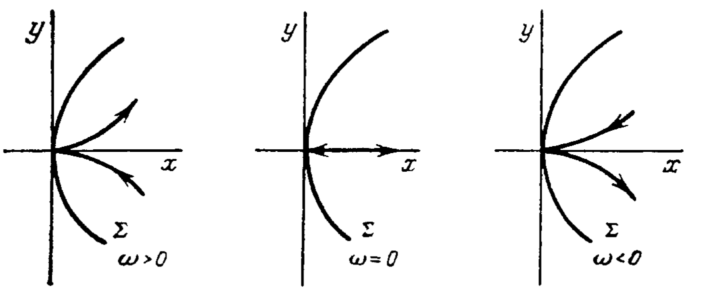

Fig. 1

[1]Historically, "Maupertuis' principle" (Theorem 1) was preceded by the simpler stationarity principle for the Hamiltonian action. "The actual content of this "principle" was not quite clear to Maupertuis. The precise formulation given in the text is due to Jacobi and his predecessors, Euler and Lagrange" (Wintner [16], 124).

Natural systems have the property of "reversibility": when $x(t)$ is any solution of the equation of motion, then $x(-t)$ is also a solution. This simple remark and the uniqueness theorem gives the following proposition.

Proposition 1. *Let $x\colon (-\varepsilon, \varepsilon) \to B$ be a motion of a natural system and let $x(0) \in \Sigma$. Then $x(t) = x(-t)$ for all $-\varepsilon < t < \varepsilon$.*

In the general irreversible case Proposition 1 is, of course, false.

Example 1. We consider a Lagrangian system ($\mathbf{R}^2 = \{x, y\}$, L) with the Lagrangian $L = (\dot{x}^2 + \dot{y}^2)/2 + \omega(x\dot{y} - y\dot{x}) + U(x, y)$. The equations of motion

$$\ddot{x} = 2\omega\dot{y} + U'_x, \quad \ddot{y} = -2\omega\dot{x} + U'_y \tag{2}$$

coincide in form with the equations of the restricted three-body problem (see [16], Ch. VI). Suppose that $(0, 0) \in \Sigma$ and that the x-axis is directed along the normal Σ into B. Let $(x, y)\colon (-\varepsilon, \varepsilon) \to B$ be a solution of (2) such that $x(0) = y(0) = 0$. Consequently, $\dot{x}(0) = \dot{y}(0) = 0$. Suppose that $x = y = 0$ is a regular point of U. Since $U'_y = 0$, we have $U'_x(0) > 0$ (taking account of the chosen direction of the x-axis). It follows from (2) that $\ddot{x}(0) = \alpha > 0$, $\ddot{y}(0) = 0$, and $\dddot{y}(0) = -2\omega\alpha$. Consequently, by Taylor's formula, we have the expansion

$$x(t) = \alpha t^2/2 + o(t^2), \quad y(t) = -2\omega\alpha t^3/3 + o(t^3).$$

Thus, close to the cusp $x = y = 0$ both branches of the trajectory are semicubical parabolas (Fig. 1). This conclusion is true, of course, also for systems of a very general form.

1.2. The geometry of a neighbourhood of the boundary.

Suppose that Σ is compact and contains no positions of equilibrium of the system ($dU|_\Sigma \neq 0$). Let $q \in \Sigma$ and $t \geqslant 0$. We denote by $x(q, t)$ the solution of the equations of motion with the following initial conditions:

$$x(q, 0) = q, \quad \left.\frac{\partial}{\partial t}\right|_{t=0} x = 0. \tag{3}$$

Our problem is to study the smooth map[(1)] $x\colon \Sigma \times [0, \varepsilon) \to B$.

Since $U: M \to \mathbf{R}$ has no critical points on Σ

$$\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} U(x(q, t)) = -2L_2^*(U'(q)) < 0, \tag{4}$$

where $L_2^*: T^*M \to \mathbf{R}$ is the function dual to the kinetic energy $L_2: TM \to \mathbf{R}$ (in the sense of the Legendre transformation). Consequently, $x\colon \Sigma \times [0, \varepsilon) \to B$ is a homeomorphism of a small neighbourhood of $\Sigma \times \{0\}$ onto some neighbourhood of Σ in B, and the inverse map is smooth outside Σ.

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(1) If B is compact, then x is defined on $\Sigma \times [0, \infty)$.

Let $s(q, t)$ be the arc length in the Jacobi metric along a geodesic $t \mapsto x(q, t)$:

$$s(q, t) = \int_0^t \left| \frac{\partial x(q, t)}{\partial t} \right| dt = 2 \int_0^t U(x(q, t))\, dt.$$

It follows from (3) and (4) that for $t = 0$

$$s = s'_t = s''_{tt} = 0, \quad s'''_{ttt} > 0.$$

By the implicit function theorem, for sufficiently small values of $r \geqslant 0$ the equation $r^3 = s(q, t)$ can be solved for t; the function $t(q, r)$ is smooth and for $r = 0$

$$t = 0, \quad t'_r > 0. \tag{5}$$

For all $q \in \Sigma$ and small $r \geqslant 0$ a smooth map $(q, r) \mapsto (q, t(q, r))$ is defined. Since its Jacobian at $r = 0$ is $t'_r > 0$ and Σ is compact, this map is a diffeomorphism in a sufficiently small neighbourhood of $\Sigma \times \{0\}$. For small $\varepsilon > 0$ we can define a map $\Sigma \times [0, \varepsilon] \to B$ by $f(q, s) = x(q, t(q, s^{1/3}))$; f maps $\Sigma \times [0, \varepsilon]$ homeomorphically to a neighbourhood of Σ in B and the restriction of f to $\Sigma \times (0, \varepsilon)$ is a diffeomorphism.

For all $0 < s \leqslant \varepsilon$ we put $W_s = f(\Sigma \times [0, s])$, $B_s = B \setminus f(\Sigma \times [0, s))$, and $\Sigma_s = f(\Sigma \times \{s\})$.

Lemma 1. *The sets W_s, B_s, and Σ_s are smooth submanifolds in B and are diffeomorphic, respectively, to $\Sigma \times [0, 1]$, B, and Σ.*

For the map $(q, r) \mapsto f(q, r^3)$ is smooth, and by (3)–(5) for $r = 0$

$$f = q, \quad f'_r = 0, \quad U''_{rr}(f(q, r^3)) < 0.$$

Proposition 2. *For small ε the set W_ε has the following properties:*

1) *geodesics in the Jacobi metric starting in Σ intersect the hyperplane $\Sigma_s \subset W_\varepsilon$ $(0 < s \leqslant \varepsilon)$ at right angles;*

2) *for any $z \in W_\varepsilon$ there is a unique geodesic γ_z starting in Σ and passing through z;*

3) *γ_z is the shortest piecewise-smooth curve joining z to Σ;*

4) *there is a $\delta > 0$ such that each geodesic in the Jacobi metric of length less than δ joining two points of W_ε lies entirely within W_ε.*

1) and 4) are analogues to well-known results of Gauss and Whitehead on Riemannian geometry. For proofs, see [17].

From this proposition it follows, in particular, that Σ_s $(s < \varepsilon)$ is the set of points of B at a distance s from the boundary. A similar geometric interpretation can be given to W_s and B_s.

1.3. Riemannian geometry of domains of possible motions with boundary. Let $a, b \in B$. We denote by Ω_{ab} the set of piecewise-smooth paths $\gamma : [0, 1] \to B$ with initial point a and end-point b. We define a function $d : B \times B \to \mathbf{R}$ by $d(a, b) = \inf\{l(\gamma): \gamma \in \Omega_{ab}\}$, where $l(\gamma)$ is the length of γ in the Jacobi metric. The non-negative function d gives a *pseudo-metric* on B, since

1) $d(a, a) = 0$ for all $a \in B$;
2) $d(a, b) = d(b, a)$ for all $a, b \in B$;
3) $d(a, b)+d(b, c) \geqslant d(a, c)$ for all $a, b, c \in B$.

We note that the pseudo-metric d is not a distance on B, since $d(a, b) = 0$ for any a, b in a single connected component of Σ. However, if $a \notin \Sigma$ then $d(a, b) = 0$ implies that $a = b$. This means that d is a distance inside B, therefore, $(B\setminus\Sigma, \langle\,,\,\rangle)$ is an (incomplete) Riemannian manifold.

The distance from $c \in B$ to the boundary Σ is defined as

$$\partial (c) = \inf_{x \in \Sigma} d(c, x).$$

If the boundary is connected, then $\partial(c) = d(c, a)$ for all $a \in \Sigma$. Also, $\partial(c) = 0$ if and only if $c \in \Sigma$. We mention that d and ∂ are continuous on $B \times B$ and B, respectively.

Proposition 3. *Let B be compact.*

α) *If Σ is connected, then for all $a, b \in B$*

$$d(a, b) \leqslant \partial(a) + \partial(b),$$

β) *if $d(a, b) < \partial(a)+\partial(b)$, then a and b can be joined by a geodesic in the Jacobi metric of length $d(a, b)$ lying entirely within $B\setminus\Sigma$.*

This result is easy to prove by standard techniques of Riemannian geometry.

Theorem 2 [4]. *If B is compact, then any $a \in B$ can be joined to some point of Σ by a geodesic line of length $\partial(a)$.*

Let $x(q, t) \in B$ be the image of (q, t) under a smooth map $\Sigma \times [0, \infty) \to B$ (see 1.2). Since the equations of motion are reversible, the following corollary holds.

Corollary.

$$\bigcup_{t \geqslant 0} \bigcup_{q \in \Sigma} x(q, t) = B.$$

Proof of Theorem 2. Let $\gamma' : [0, 1/2] \to B$ be a shortest geodesic joining $\gamma'(0) = a$ to the hypersurface Σ_ε. Such a curve exists and is orthogonal to Σ_ε at $\gamma'(½)$. By Proposition 3 there is a geodesic $\gamma'' : [½, 1] \to B$ of length joining $\gamma''(½) = \gamma'(½)$ to the boundary Σ. The curve $\gamma : [0, 1] \to B$, which coincides with $\gamma'(\gamma'')$ in $[0, ½]$ $([½, 1])$, is obviously a smooth geodesic of length $\partial(a)$, where $\gamma(0) = a$ and $\gamma(1) \in \Sigma$.

Theorem 2 can be regarded as an analogue of the Hopf-Rinow theorem (see [18]) of Riemannian geometry. In contrast to the Riemannian case, here even for compact B not all pairs of points can be joined by a geodesic in the Jacobi metric.

Example 2. We consider the oscillations of a plane harmonic oscillator described by the equations $\ddot{x} = -x$, $\ddot{y} = -y$, with total energy $h = 1/2$. In this problem B is the unit circle $x^2+y^2 \leqslant 1$. It can be shown that the set of points of the unit circle that can be reached from a point $(x, y) = (a, 0)$ with initial velocity $|v| = \sqrt{(1-a^2)}$, is given by the inequality $x^2+y^2/(1-a^2) \leqslant 1$ (see Fig. 2). When $a \to 0$ (or $a \to 1$), this "accessible set"[1] tends to the whole of B (or to the segment $y = 0$, $-1 \leqslant x \leqslant 1$).

In the irreversible case Theorem 2 is false.

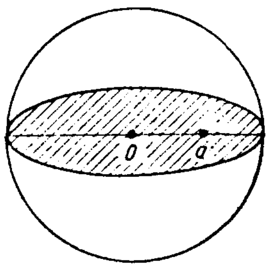

Fig. 2

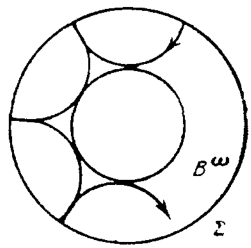

Fig. 3

Example 3. We consider a plane harmonic oscillator under the action of additional gyroscopic forces:

$$\ddot{x} = -2\omega\dot{y} - x, \quad \ddot{y} = 2\omega\dot{x} - y. \tag{6}$$

These equations describe, in particular, the small oscillations of a Foucault pendulum (see [25]). In this problem B is again the circle $x^2+y^2 \leqslant 1$. Let B^{ω} be the set of points of B that can be reached from Σ by moving along trajectories of (6). In polar coordinates r, φ the equations (6) take the following form:

$$\ddot{r} = r(\dot{\varphi}(\dot{\varphi} - 2\omega) - 1), \quad (r^2\dot{\varphi})^{\boldsymbol{\cdot}} = (\omega r^2)^{\boldsymbol{\cdot}}.$$

The second equation can be integrated: $r^2\dot{\varphi} = \omega r^2 + c$. Since $\dot{r} = \dot{\varphi} = 0$, $r = 1$ for $t = 0$, we see that $c = -\omega$. After substituting $\dot{\varphi} = (1 - r^{-2})\omega$ in the first equation we obtain a system with one degree of freedom:

$$\ddot{r} = -(1+\omega^2)\,r + \frac{\omega^2}{r^3}.$$

From the energy integral

$$\frac{\dot{r}^2}{2} + (1+\omega^2)\frac{r^2}{2} + \frac{\omega^2}{2r^2} = \frac{1}{2} + \omega^2$$

(1)The general definition of an accessible set was given by Tatarinov in [19].

it follows that B^ω is given by the inequalities

$$\frac{\omega^2}{1+\omega^2} \leqslant r^2 \leqslant 1.$$

Consequently, B^ω for $\omega \neq 0$ does not coincide with B; as $\omega \to 0$, B^ω tends to B and as $\omega \to \infty$, B^ω degenerates to the boundary $\Sigma = \{x^2 + y^2 = 1\}$. The trajectories of (6) starting on Σ are depicted in Fig. 3. For almost all ω they fill B^ω densely. If a trajectory of (6) passes through the origin then $c = 0$ and, consequently, $\dot{\varphi} = \omega$. In this case the point performs harmonic oscillations with frequency $\sqrt{(1+\omega^2)} > 1$ along a segment of length $2/\sqrt{(1+\omega^2)} < 2$ passing through the origin and uniformly rotating with constant angular velocity ω. The presence of these motions is a characteristic property of Foucault's pendulum.

In the general irreversible case we denote by B^+ the closed set of points of B where $4L_0L_2 \geqslant L_1^2$. If we exclude the degenerate case when $L_1 = 0$ at some point of Σ, then $B^+ \subset B \setminus \Sigma$. Inside B^+ the integrand in the functional of the abbreviated action F^+ is positive definite. This property holds simultaneously for mechanical systems with Lagrangians $L_\pm = L_2 \pm L_1 \pm L_0$. If $x(t)$ is a solution of $[L_+] = 0$, then $x(-t)$ is one of $[L_-] = 0$ and vice versa.

It can happen that B^+ is empty. In this case we can proceed as follows. Let $L_1 = a(x) \cdot \dot{x}$ and $x_0 \in B \setminus \Sigma$. We change L_1 locally into $L_1 - \hat{L}_1$, where $\hat{L}_1 = a(x_0) \cdot \dot{x}$. Since $\hat{L}_1$ is closed, the Lagrangian equation $[L] = 0$ remains unchanged. For the new Lagrangian $4L_0L_2 > L_1^2$ holds in a small neighbourhood of x_0, since $L_1 \equiv 0$ at $x = x_0$. This remark allows us to vary the form and situation of the domain B^+.

We decrease B^+ by removing an ε-neighbourhood (for example, in the Jacobi metric) of ∂B^+. The remaining set is denoted by B_ε^+.

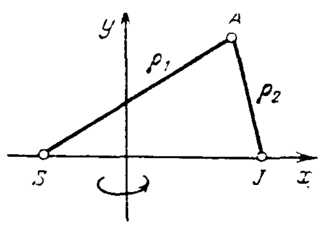

Fig. 4

***Proposition* 4.** *Let B be compact and let B_ε^+ be non-empty for some $\varepsilon > 0$. Then:*

1) *for any $a \in B_\varepsilon^+$ there is a solution x: $[0, \tau] \to B_\varepsilon^+$ of Lagrange's equation $[L] = 0$ such that $x(0) = a$ and $x(\tau) \in \partial B_\varepsilon^+$;*

2) *for any $a \in B_\varepsilon^+$ there is a solution y: $[0, \tau] \to B_\varepsilon^+$ such that $y(0) \in \partial B_\varepsilon^+$ and $y(\tau) = a$.*

Proof. The curve $x(t)$ (or $y(-t)$) attains a minimum of the action functional F^* corresponding to the Lagrangian L_+ (or L_-) in the set of piecewise-smooth curves joining a to points of the boundary B_ε^+.

Remark. In contrast to Theorem 2, the constant ε in Proposition 4 cannot be put equal to zero (even when $B^+ \subset B \setminus \Sigma$).

Example 4. The motion of an asteroid in the restricted three-body problem is described by (2) where we have to put $\omega = 1$ and

$$U = \frac{x^2+y^2}{2} + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}; \quad \rho_1 = \sqrt{(x+\mu)^2+y^2}, \quad \rho_2 = \sqrt{(x-1+\mu)^2+y^2}.$$

In this problem the Sun and Jupiter, with masses $1-\mu$ and μ, rotate with unit angular velocity along circular orbits, with radii μ and $1-\mu$, around their common centre of mass, and an asteroid, a body of negligible mass, moves in the plane of the ecliptic undergoing attractions from the Sun and Jupiter (Fig. 4); for details, see [16] Ch. VI. The domain B (the so-called Hill domain in celestial mechanics) is given by $U \geqslant -h$. If L_1 has the "standard" form $x\dot{y} - y\dot{x}$, then B^+ is the set

$$\left\{\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \geqslant -h\right\},$$

which is the domain of possible motions in the problem of two fixed centres (fixed Sun and Jupiter attracting the asteroid according to the law of universal gravitation).

§2. Periodic trajectories of natural mechanical systems

2.1. Rotations and librations.

A solution $x : \mathbf{R} \to L$ of Lagrange's equation $[L] = 0$ is periodic if $x(t+\tau) = x(t)$ for all $t \in \mathbf{R}$ and some $\tau > 0$. The trajectory of a periodic solution is always closed. We are interested in the existence of closed trajectories for fixed values of the total energy h. We assume that h is a regular value (to exclude trivial periodic solutions, namely, equilibria).

Proposition 5. *A closed trajectory γ of a periodic solution $x : \mathbf{R} \to B$ with zero energy*

1) *either does not intersect Σ,*
2) *or has precisely two points in common with Σ.*

To each trajectory of the first type there correspond two distinct periodic solutions (rotations relative to γ in opposite directions), and to one of the second type a unique periodic solution (oscillatory motion between the boundary points of γ). Periodic motions of the first type are called rotations and of the second type librations. It follows from Proposition 1 that if the trajectory of some solution $x : \mathbf{R} \to B$ has two points in common with Σ, then there are no other common points, and $x(\cdot)$ is a libration.

If $\Sigma = \varnothing$, then the question of the existence of periodic motions reduces to that of the presence of closed geodesics of a Riemannian manifold $(M, \langle , \rangle)$. This classic problem of Riemannian geometry has been well studied (at least for compact M). If M is not simply-connected then, as Hadamard showed in 1898, each closed curve that is not homotopic to zero can be deformed into a closed geodesic of minimal length in its free homotopy class. This remark gives a lower bound on the number of distinct closed geodesics on a non-simply-connected manifold. The problem of the existence of periodic geodesics in the case of a simply-connected M is much more complicated. In 1905 Poincaré established the existence of such curves on the convex two-dimensional sphere[1]. Later this result was extended by Birkhoff to the case of an arbitrary many-dimensional Riemannian sphere. Lyusternik and Shnirel'man proved (in 1929) that there are three non-self-intersecting closed geodesics on the two-dimensional sphere. With certain additional restrictions an analogous result holds in the many-dimensional case: if the Gaussian curvature K of the sphere S^n (at all points and in all two-dimensional directions) satisfies $K_0/4 < K \leqslant K_0$, for some $K_0 > 0$, then there are n non-self-intersecting closed geodesics on S^n (Klingenberg). The existence of a closed geodesic on any compact manifold was established by Lyusternik and Fet (1951). For certain simply-connected manifolds it has even been proved that there are infinitely many distinct closed geodesics (Gromoll and Meyer). At present it is not completely clear whether this result holds in the general case of a simply-connected manifold. A detailed survey of the present state of these questions can be found in the book by Klingenberg [20].

For a non-empty Σ the situation as regards the existence of periodic trajectories is clarified by other means. A good idea in this case is given by the following example.

Example 5. We consider the oscillations of a "poly-harmonic" oscillator, described by the equations $\ddot{x}^s + \omega_s^2 x^s = 0$ $(1 \leqslant s \leqslant n)$ with rationally independent frequencies $\omega_1, \ldots, \omega_n$. The domain of possible motions B with total energy h is the ellipsoid $\sum \omega_s^2 x^{s^2} \leqslant 2h$. For any $h > 0$ the equations of motion have precisely n periodic oscillations, namely, librations, whose trajectories coincide with the principal axes of this ellipsoid. It is worth stressing that there are no rotations and that the number of periodic trajectories for a fixed value of the total energy is finite. If the frequencies $\omega_1, \ldots, \omega_n$ are rationally commensurable, then the number of librations may increase. For example, in the case $n = 2$ when ω_1/ω_2 is rational, there is a

[1] Poincaré proposed two approaches to the solution of this problem. The first was based on the principle of analytic continuation of periodic trajectories (see also [34]). The second was purely variational: among the closed non-self-intersecting curves that divide the sphere into two halves with equal total curvature there is a curve of minimal length; this is the required closed geodesic.

libration trajectory through each point of Σ. To prove this we have to take account of the fact that if ω_1/ω_2 is rational, then all trajectories are closed and use Proposition 5. Note the special case $\omega_1 = 1$, $\omega_2 = n \in \mathbf{N}$. The libration energies h are given by $x^1 = x_0^1 \cos t$, $x^2 = x_0^2 \cos nt$, where $x_0^1 = \sqrt{(2h)} \sin \alpha$, $x_0^2 = \sqrt{(2h)}(\cos \alpha)/n$, and α is an arbitrary constant. Let $2h = n^2 + 1$; then for some α the values of x_0^1 and x_0^2 are 1 and the trajectory of the corresponding libration in $\mathbf{R}^2 = \{x^1, x^2\}$ coincides with part of the graph of the Chebyshëv polynomial T_n.

2.2. Librations in non-simply-connected domains of possible motions. For any group π we denote by $r(\pi)$ the least number of its generators. Let B/Σ be the topological space obtained from B by contracting Σ to a point, and $\pi(B/\Sigma)$ its fundamental group.

***Theorem* 3.** *Suppose that B is compact and that there are no equilibrium positions on Σ. Then the number of distinct librations in B is at least $r(\pi(B/\Sigma))$.*

Remark. This number is not less than the first Betti number of B modulo Σ.

Corollary. *If Σ has n connected components then the number of librations in B is at least $n-1$. In addition, for each connected component of Σ there is a libration with an end-point in this component, and the trajectories of these librations have no self-intersections (Fig. 5).*

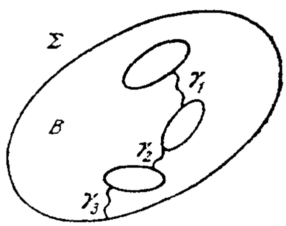

Fig. 5

For in this case $\pi(B/\Sigma)$ contains a free group with $(n-1)$ generators.

Theorem 3 was proved by Bolotin and the author in [17]; it is analogous to the standard theorem of Hadamard on minimal closed geodesics on a non-simply-connected Riemannian manifold.

Let us give an idea of the proof of the corollary. We may suppose that B is a closed submanifold with boundary of a compact Riemannian manifold M whose metric on $B \setminus W_\varepsilon$ coincides with the Jacobi metric. Let $\Sigma_\varepsilon^1, \ldots, \Sigma_\varepsilon^n$ be the connected components of Σ_ε and d_{ij} $(i < j)$ the distance between Σ_ε^i and Σ_ε^j. We fix i; among the numbers d_{is} $(i \neq s)$ there is a smallest. This is realized by a non-self-intersecting geodesic γ_i of length d_{is} lying entirely in $B \setminus W_\varepsilon$ and orthogonal to Σ_ε at both ends. The geodesic γ_i can be extended to a libration periodic solution using Proposition 2 (compare with the proof of Theorem 2). The number of such distinct "minimal" librations is obviously at least $n-1$.

Example 6 ([17]). We consider the problem of the existence of librations of a plane n-linked mathematical pendulum (Fig. 6). Let $l_1, \ldots, l_n$ be the lengths of the links, counting from the point of suspension, $P_1, \ldots, P_n$ the weights of the corresponding material points, and $\vartheta^1, \ldots, \vartheta^n$ the angles formed by the links with the vertical. The configuration space M is an n-dimensional torus $\mathbf{T}^n = \{\vartheta^1, \ldots, \vartheta^n \bmod 2\pi\}$, and the potential energy has the form

$$V = -\sum_{i=1}^{n} a_i \cos\vartheta^i, \quad a_i = l_i \sum_{j=i}^{n} P_j.$$

The set of critical points of $V : \mathbf{T}^n \to \mathbf{R}$ is in one-to-one correspondence with the set of all subsets of $\Lambda = \{1, 2, \ldots, n\}$; the index of the critical point corresponding to a subset $I \subset \Lambda$ is equal to the number of elements in I, and the critical value is

$$h_I = \sum_{i\in I} a_i - \sum_{i\notin I} a_i.$$

Let h be a non-critical value of the potential energy and $|h| < \sum_\Lambda a_i$. In this case the domain $B \subset \mathbf{T}^n$ has a non-empty boundary Σ. We put $\hat{B} = \overline{\mathbf{T}^n \setminus B}$. Since $B/\Sigma = \mathbf{T}^n/\hat{B}$, we see that $\pi(B/\Sigma) = \pi(\mathbf{T}^n/\hat{B})$. We put $r = r(\pi(\mathbf{T}^n/\hat{B}))$ and $\hat{r} = r(\pi(\hat{B}))$. Let $k \leqslant n$ be the number of critical points of the function $V : \mathbf{T}^n \to \mathbf{R}$ in B with index $(n-1)$. We claim that $r = k$.

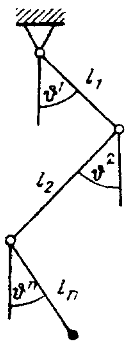

Fig. 6

For $\mathbf{T}^n/\hat{B}$ (or $\hat{B}$) is homotopy equivalent to a cell complex with k (respectively, $n-k$) one-dimensional cells. Therefore $r \leqslant k$, $\hat{r} \leqslant n-k$. Since $\pi(\hat{B})$ and $\pi(\mathbf{T}^n/\hat{B})$ generate $\pi(\mathbf{T}^n)$, we see that $n \leqslant r+\hat{r} \leqslant k+(n-k) = n$. Therefore, $r = k$.

Applying Theorem 3 we obtain the following result: if $h \neq h_I$ for all $I \subset \Lambda$ and $|h| < \Sigma_\Lambda a_i$, then the number of librations with total energy h is not less than the number of indices i such that $a_1 + \ldots + a_{i-1} + a_{i+1} + \ldots + a_n < h$. Depending on the value of h this lower bound on the number of librations varies from 0 to n. In §2.4 it will be shown how this bound can be improved using symmetry properties.

Theorem 3 on shortest librations has the following sharper form due to Bolotin.

***Theorem* 4.** *Suppose that B is compact and that there are no positions of equilibrium on Σ. Then there are at least $r(\pi(B/\Sigma))$ distinct unstable librations in B with real characteristic multipliers.*

The proof consists simply of the verification that all characteristic multipliers of the shortest librations, whose existence is guaranteed by Theorem 3, are real.

Example 7. We consider the motion of a material point in a central field with potential $V = r+r^{-1}$. For $h > 2$ the domain B is an annulus, and obviously, through each point of its boundary there is a librational trajectory. All these librations are shortest; they are degenerate, since all their characteristic multipliers are zero. The instability of the librations is easily deduced with the help of the area integral: $r^2\dot{\varphi} =$ const.

In the general case the characteristic multipliers are non-zero, therefore, the shortest librations are (orbitally) unstable in the linear approximation. In addition, since these librations are hyperbolic periodic solutions, there are families of trajectories asymptotically approaching the trajectories of the shortest librations as $t \to \pm\infty$.

Let $A(\gamma)$ denote the set of points in B that are on trajectories asymptotic to γ. An idea of the form and situation of $A(\gamma)$ is given by the following example.

Example 8 (Bolotin). Let $M = S^1\{x \bmod 2\pi\} \times \mathrm{R}\{y\}$ and $L = (\dot{x}^2+\dot{y}^2)/2 + \cos x - y^2/2$. For $h > 1$ the domain B is diffeomorphic to the ring $|y| \leqslant \sqrt{(2(h+\cos x))}$ and the curve $x = \pi$, $y = \sqrt{(2(h-1))}\cos t$ is a shortest libration with energy h. The Lagrange equations $[L] = 0$ have two first integrals $\dot{x}^2/2 - \cos x$ and $\dot{y}^2+y^2$. With their help it is easy to show that $A = \{|y| \leqslant \sqrt{(2(h-1))}\}$. Note that A is not equal to B and that $A \cap \Sigma$ consists of the ends of the trajectories of the shortest librations.

2.3. Librations in simply-connected domains and Seifert's conjecture. The first general result on librations of natural systems is due to Seifert, who proved in [1] that there are librations when B is diffeomorphic to an n-dimensional disc.

***Theorem* 5** (Bolotin). *If B is compact and Σ contains no critical points of the potential, then there is at least one libration in B.*

Continuing the analogy with Riemannian geometry we may regard the theorems of Seifert and Bolotin as corresponding to the results of Birkhoff and Lyusternik-Fet on closed geodesics on the n-dimensional sphere and arbitrary simply-connected manifolds.

The proof of Theorem 5 is based on the following result:

***Lemma* 2.** *There is an $l > 0$ such that for all $0 < \varepsilon \leqslant \delta$ there is in $\Lambda_\varepsilon = \overline{B \setminus W_\varepsilon}$ a geodesic in the Jacobi metric of length less than l with end-points on Σ_ε and intersecting Λ_δ.*

We deduce Theorem 5 from this. For all s in $(0, \delta)$ we denote by $\gamma_s: [a_s, b_s] \to \Lambda_s$ the geodesic of Lemma 2. We suppose that γ_s has the natural parametrization, also that $a_s < 0 < b_s$ and $\gamma_s(0) \in \Lambda_\delta$. Since Λ_δ is compact, there is a sequence $s_n \to 0$ such that

$$\lim_{n\to\infty} \gamma_{s_n}(0) = x \in \Lambda_\delta, \quad \lim_{n\to\infty} \dot{\gamma}_{s_n}(0) = v.$$

Let $\gamma: (a, b) \to B$ be the unique maximal geodesic in the Jacobi metric for which $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Clearly, γ is the trajectory of a libration in B whose length does not exceed l.

Lemma 2 is proved by methods of Morse theory. We choose a small $\delta > 0$ and let $0 < s < \delta$. We introduce the space of piecewise-smooth curves $\gamma: [0, 1] \to M$ such that $\gamma(0), \gamma(1) \in \Sigma_\varepsilon$; we denote this space by Ω. Let Γ be the subspace of Ω consisting of the curves that do not intersect the interior of Λ_s. On M we can indicate a family of smooth functions V_s, $0 < s \leqslant \varepsilon$, such that V_s coincides with V in Λ_s, $V_s \geqslant V_\varepsilon$ on M, and $\sup\limits_M V_s < h$. For all $s \in (0, \varepsilon]$ we specify a metric $\langle , \rangle_s$ on M, the Jacobi metric defined by the potential V_s and the energy h. Finally, we define the action functional $F_s: \Omega \to \mathbf{R}$ by

$$F_s(\gamma) = \int_0^1 \langle \dot{\gamma}, \dot{\gamma} \rangle_s \, dt.$$

The critical points of F_s are precisely the geodesics in the metric $\langle , \rangle_s$ that are orthogonal to Σ_s at both ends.

For any $a > 0$ we put $\Omega_s^a = \{\gamma \in \Omega: F_s(\gamma) \leqslant a\}$ and $\Gamma_s^a = \Gamma \cap \Omega_s^a$.

***Lemma* 3.** *If F_s, $0 < s \leqslant \varepsilon$, has no critical points in $\Omega_s^a \setminus \Gamma_s^a$, then Γ_s^a is a deformation retract of Ω_s^a.*

The idea of the proof is to shift Ω_s^a "down" onto Γ_s^a along integral curves of the gradient field of F_s. The main feature is the use of the convexity of W_δ: the relevant "curves of steepest descent" do not leave Γ_s^a.

Lemma 2 is deduced from Lemma 3 and the following topological fact: since B/Σ is not contractible, Γ_s^a for sufficiently large $a > 0$ is not a deformation retract of Ω_s^a. A detailed proof of Theorem 5 is in [21].

Example 9 [3]. We consider the rotation of a rigid body in an axisymmetric force field with potential V. For zero values of the constant of kinetic momentum this problem reduces to the investigation of a natural system with two degrees of freedom on the sphere. Theorems 3 and 5 and the results of Morse theory imply the following assertion: for all non-critical values $h > \min V$ this system has periodic solutions of energy h. If $h > \max V$, then by the Lyusternik-Shnirel'man theorem there are at least three distinct non-self-intersecting periodic trajectories on the Poisson sphere.

Remark. The problem of periodic trajectories of "Lorentz" Lagrangian systems with Lagrangian $(S\dot{x}, \dot{x})/2 - V(x)$, where $(\, , \,)$ is the standard scalar product in $\mathbf{R}^n$ and S is a symmetric non-degenerate linear operator with one negative eigenvalue, is discussed in [39]. Let $\Sigma = \{y \in \mathbf{R}^n: (Sy, y) < 0\}$ be a cone in $\mathbf{R}^n$. If $x(\cdot)$ is a motion with zero total energy starting in $C = \{V(x) > 0\}$, then $\dot{x} \in \Sigma$. Since Σ consists of two connected components, a transition from one component to the other (a change of the "direction" of motion) can happen only on the boundary of C. If x hits ∂C, then Proposition 1 is valid: the point moves along the same trajectory in the opposite direction. In [39] it was proved that under the assumption of compactness and convexity of C there are librations provided that there are no critical points of V on the boundary of C. The proof is based on an application of topological fixed point theorems for smooth maps.

In connection with Theorem 5 it is natural to ask for lower bounds on the number of distinct librations when B/Σ is simply-connected. Example 5 shows that a universal bound cannot exceed the dimension of B. The Seifert conjecture asserts that there are n distinct librations when B is diffeomorphic to an n-dimensional disc D^n. Up to now the conjecture has been neither proved nor disproved. We quote a result, due to Bolotin, in favour of the Seifert conjecture.

Suppose that the domain of possible motions B is diffeomorphic to D^n and that $(D^n, S^{n-1}) \to (B, \Sigma)$ is a continuous surjective map. For any two points $x, y \in S^{n-1}$ we define a continuous curve $f_{x,y}$: $[0, 1] \to B$ by $f_{x,y}(t) = f((1-t)x + ty)$, $0 \leqslant t \leqslant 1$. We assume that f is smooth enough so that $f_{x,y}$ for all $x, y \in S^{n-1}$ is piecewise-smooth. The abbreviated action F^* is defined on such curves. We put

$$S = \inf_{f} \sup_{x, y \in S^{n-1}} F^*(f_{x,y}).$$

***Theorem* 6.** *Suppose that for any libration γ in $B \simeq D^n$, $2F^*(\gamma) > S$. Then there are n distinct librations $\gamma_1, \ldots, \gamma_n$ in B such that $S/2 < F^*(\gamma_1) \leqslant \ldots \leqslant F^*(\gamma_n) = S$.*

Example 10. We continue with the discussion of Example 5. Let $\omega_1 \geqslant \ldots \geqslant \omega_n > 0$ be the frequencies of a poly-harmonic oscillator. As we have already seen, this problem always has n distinct librations of energy h:

$$\gamma_1\colon\ x^1 = \sqrt{2h}/\omega_1 \cos\omega_1 t, \qquad x^i = 0 \quad (i > 1),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\gamma_n\colon\ x^i = 0 \ \ (i < n), \qquad x^n = \sqrt{2h}/\omega_n \cos\omega_n t.$$

It is easy to calculate that $F^*(\gamma_i) = \pi h/\omega_i$; thus, $F^*(\gamma_1) \leqslant \ldots \leqslant F^*(\gamma_n)$. Here $S = F^*(\gamma_n)$ and the condition of Theorem 6 is equivalent to $2\omega_n > \omega_1$.

Remark. A related problem of the existence of periodic solutions of Hamilton's equations in $\mathbf{R}^{2n}$ with a convex Hamiltonian H was discussed in [10]. It was proved that if for some $a > 0$

$$a \mid z \mid^2 < H(z) < 2a \mid z \mid^2,$$

then on each level set $H(z) = h$, $h > 0$, the Hamiltonian system has at least n distinct closed trajectories. For a poly-harmonic oscillator with frequencies $\omega_1 \geqslant \ldots \geqslant \omega_n > 0$ these inequalities give the same condition $2\omega_n > \omega_1$, since by the substitution $q_i = \sqrt{(\omega_i x^i)}$, $p_i = \dot{x}^i/\sqrt{\omega_i}$ the equations of the vibrations of the oscillator reduce to Hamiltonian form with the Hamiltonian

$$H = \frac{1}{2} \sum \omega_i (p_i^2 + q_i^2).$$

2.4. Periodic oscillations of a many-linked pendulum [22].
Here we obtain an estimate of the number of distinct periodic motions (both librations and rotations) of given energy of the compound pendulum of Example 6. The configuration space of this system is the n-dimensional torus $\mathbf{T}^n = \{\vartheta^1, \ldots, \vartheta^n \bmod 2\pi\}$ where $\vartheta^1, \ldots, \vartheta^n$ are the angles formed by the links with the vertical. We may assume that the configuration space is the covering space $\mathbf{R}^n = \{\vartheta^1, \ldots, \vartheta^n\}$ and that the Lagrangian L is a function on $T\mathbf{R}^n$ that is 2π-periodic in ϑ^i. The positions of equilibrium are the points of $\mathbf{R}^n$ of the following form: $a = (m_1\pi, \ldots, m_n\pi)$, where the m_s are integers. It is easy to see that the Lagrangian admits a reflection of $\mathbf{R}^n$ relative to a position of equilibrium, that is, the map Λ_a: $\vartheta \to -\vartheta + 2a$.

***Lemma* 4.** *If a trajectory of some motion $\vartheta(t)$ passes through a ($\vartheta(0) = a$), then this curve is invariant under the reflection Λ_a (that is, $\vartheta(-t) = \Lambda_a\vartheta(t) = -\vartheta(t) + 2a$). In particular, $\dot{\vartheta}(-t) = \dot{\vartheta}(t)$.*

***Lemma* 5.** *Let $b \in \mathbf{R}^n$ be another equilibrium ($a \neq b$). If a trajectory of a motion $\vartheta(t)$ contains points a and b, then:*
1) *there is a $\tau > 0$ such that $\vartheta(t + \tau) = \vartheta(t) + 2(b - a)$ for all $t \in \mathbf{R}$,*
2) *$\dot{\vartheta}(t) \neq 0$ for all $t \in \mathbf{R}$.*

Let h_- (h_+) be the smallest (largest) value of the potential energy $V(\vartheta)$.

***Proposition* 6.** *Let h be a non-critical value of the potential in the interval (h_-, h_+). Through each critical point of V in the interior of $B = \{V \leqslant h\} \subset \mathbf{T}^n$ there passes at least one libration trajectory. The librations passing through different critical points are distinct.*

Corollary. *The number of distinct librations in B is at least equal to the number of positions of equilibrium of the pendulum in the interior of B.*

Depending on h this lower bound of the number of librations of energy h varies from 1 to $2^n - 1$. This estimate strengthens the result mentioned in Example 6. True, the estimate in Example 6 holds even when the potential has no symmetry properties.

Proof of Proposition 6. Let $a' \in B \subset \mathbf{T}^n$ be a position of equilibrium of the pendulum. Since h is non-critical, by Theorem 2 there is a motion $\gamma : [0, \tau] \to B$ such that $\gamma(0) = a'$ and $\gamma(\tau) \in \Sigma$. According to Lemma 4, $\gamma : \mathbf{R} \to B$ is the required libration whose trajectory contains a'. The librations passing through different critical points are distinct because otherwise (by Lemma 5) the velocity of the motion can never vanish.

We now consider the case $h > h_+$. Since $\Sigma = \varnothing$, periodic solutions can only be rotations. We examine the question of the existence of periodic rotations of an n-linked pendulum under which the k-th link makes N_k complete revolutions during the period of rotation. We call such motions rotations of type $]N_1, \ldots, N_n[$.

Proposition 7. *For any fixed integers* $N_1, \ldots, N_n$ *and any* $h > h_+$ *there are* 2^{n-1} *distinct periodic rotations of type* $]N_1, \ldots, N_n[$ *with total energy* h *whose trajectories on* $\mathbf{T}^n$ *pass through pairs of critical points of* V.

Proof. Obviously, we may assume that $N_1, \ldots, N_n$ are relatively prime. We consider a pair of critical points a' and a'' of V in $\mathbf{R}^n = \{\vartheta\}$ whose ϑ^k-coordinates are different from πN_k. These points cover distinct points b' and b'' on $\mathbf{T}^n$. We join b' and b'' by a shortest geodesic in the Jacobi metric on $\mathbf{T}^n$. To this geodesic there corresponds a motion $\gamma : \mathbf{R} \to \mathbf{T}^n$ such that $\gamma(t') = b'$ and $\gamma(t'') = b''$, $t'' > t'$. Suppose that the curve $\vartheta : \mathbf{R} \to \mathbf{R}^n$ covers γ and that $\vartheta(t') = a'$, $\vartheta(t'') = a''$. By Lemma 2 there is a $\tau > 0$ such that

$$\vartheta(t + \tau) - \vartheta(t) = 2(a'' - a') = (2\pi N_1, \ldots, 2\pi N_n).$$

Consequently, the motion $\gamma : \mathbf{R} \to \mathbf{T}^n$ is periodic of type $]N_1, \ldots, N_n[$ with period τ. Since $N_1, \ldots, N_n$ are relatively prime, τ is the least period of γ. From this remark and Lemma 5 it is easy to deduce that the trajectory of γ on $\mathbf{T}^n$ contains no points of equilibrium other than b' and b''.

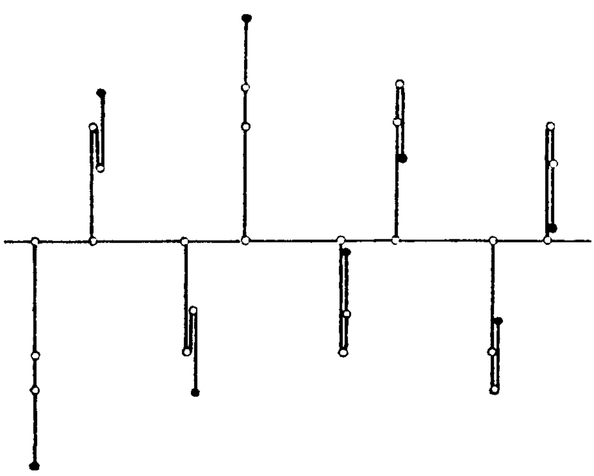

Fig. 7

In Fig. 7 are depicted four pairs of equilibrium positions of a three-linked pendulum that are taken for distinct periodic solutions of type]1, 2, 3[.

In conclusion we show that under certain conditions periodic rotations of the pendulum exist even for values $h < h_+$. To see this we consider a double pendulum with identical rod lengths l, masses m_1 and m_2, g being the acceleration of free fall. The Lagrangian is

$$L = \frac{1}{2}(m_1+m_2) l^2\dot{\vartheta}^{1^2} + \frac{m}{2} l^2\dot{\vartheta}^{2^2} + m_2 l^2\dot{\vartheta}^1\dot{\vartheta}^2\cos(\vartheta^1-\vartheta^2) + \\ + m_1 gl\cos\vartheta^1 + m_2 gl(\cos\vartheta^1+\cos\vartheta^2).$$

We consider the case when h is close to h_+. Fixing the value of m_1 we let m_2 tend to zero. For sufficiently small m_2 the distance between $a = (0, 0)$ and $b = (0, \pi)$ is less than the sum of the distances from these points to the boundary Σ. For $d(a, b)$ does not exceed the length of the segment $\{\vartheta^1 = 0, 0 \leqslant \vartheta^2 \leqslant \pi\} \subset \mathbf{R}^2$, which is

$$\sqrt{m_2}\, l \int_0^\pi \sqrt{h+m_1 gl + m_2 gl(1+\cos\vartheta^2)}\, d\vartheta^2.$$

This quantity tends to zero as $m_2 \to 0$. Since here B is near to $\{h + m_1 gl\cos\vartheta^1 \geqslant 0\}$,

$$\lim_{m_2\to 0} \partial(a) = \frac{\sqrt{m_1}\, l}{2} \oint \sqrt{h+m_1 gl\cos\vartheta_1}\, d\vartheta_1 > 0.$$

Hence, $d(a, b) < \partial(a) + \partial(b)$ for small m_2. By Proposition 3 there is a shortest geodesic in the Jacobi metric in the interior of B and joining a to b. To this geodesic there corresponds a solution of the equations of motion with total energy h. Since a and b are positions of equilibrium, by Lemma 5 this solution is a periodic rotation.

§3. Periodic trajectories of irreversible systems

3.1. Systems with gyroscopic forces and many-valued functionals.
So far we have considered the situation when the "seminatural" Lagrangian $L = L_2 + L_1 + L_0$ is a single-valued function on the tangent bundle TM. In particular, the 1-form $\omega \equiv L_1$ is defined and single-valued everywhere on M. Consequently, its exterior differential $\Omega = d\omega$, the 2-form of gyroscopic forces, is exact. It is useful to generalize this situation by considering mechanical systems with a closed (but not necessarily exact) form of gyroscopic forces.

Example 11. The motion of a charge in the Euclidean plane $\mathbf{R}^2 = \{x, y\}$ in a magnetic field (directed along the z-axis) with intensity $H(x, y)$ can be described by the equations $\ddot{x} = -H\dot{y}$, $\ddot{y} = H\dot{x}$. The form of gyroscopic forces Ω is, obviously, $H\, dx \wedge dy$. It is, of course, exact. If, for example, $H = \text{const}$, then $\omega = H(y dx - x dy)/2$. We consider the special case when the magnetic field $H(x, y)$ is 2π-periodic in x and y. In this case for the configuration space we can take the two-dimensional torus $\mathbf{T}^2 = \{x, y \bmod 2\pi\}$ with the flat metric. The form Ω is exact if and only if the total flux of the magnetic field

$$\overline{H} = \int_0^{2\pi}\int_0^{2\pi} H\, dx \wedge dy$$

vanishes.

Example 12. We consider the motion of a charge on the surface of the unit sphere $\langle r, r\rangle = 1$ in $\mathbf{R}^3 = \{r\}$. We assume that the magnetic field is uniform in size and directed orthogonal to the surface of the sphere. The equation of motion can be presented by means of the Lagrange multiplier

$$\ddot{r} = H(\dot{r} \times r) + \lambda r, \quad \langle r, r\rangle = 1; \quad H = \text{const.}$$

Hence $\lambda = -\langle \dot{r}, \dot{r}\rangle$. Since the total energy $E = \langle \dot{r}, \dot{r}\rangle/2$ is preserved, $\lambda(t) = -2E = \text{const}$. It can be shown that for a fixed value of E the trajectories of this equation are circles of radius ρ, where

$$\rho^2 = \frac{2E/H^2}{1+2E/H^2}. \tag{7}$$

In Example 11 the "Larmor radius" is $\rho = \sqrt{(2E)}/H$. The form of gyroscopic forces is $\Omega = H d\sigma$, where $d\sigma$ is the element of area on the unit sphere. It is not exact, since the total flux of the magnetic field through the sphere is $4\pi H \neq 0$.

Example 13. The rotation of a rigid body with a fixed point in an axisymmetric field can be described by the system of Euler-Poisson equations (in mobile space)

$$\dot{M} = M \times \omega + e \times V', \quad \dot{e} = e \times \omega.$$

Here $M = I\omega$ is the kinetic momentum of the rigid body, ω its angular velocity, I the inertia tensor, e the unit vector along the axis of symmetry of the field, and $V(e)$ the potential. It is not hard to show that in a fixed level of the area integral

$$\{(M, e) \in \mathbf{R}^6\colon \langle M, e\rangle = c, \ \langle e, e\rangle = 1\}$$

there arises a system with gyroscopic forces. The two-dimensional sphere S^2 serves as configuration space, and the form Ω_c is not exact for $c \neq 0$, since

$$\int_{S^2} \Omega_c = 4\pi c. \tag{8}$$

A related example is provided by Kirchhoff's problem of the motion of a rigid body in an unbounded ideal fluid (see [23]). We fix the constant Kirchhoff integrals $\langle e, e\rangle = p \neq 0$, $\langle M, e\rangle = c$. The corresponding integral level is a four-dimensional manifold diffeomorphic to the tangent bundle of S^2. On S^2 there arises a system with gyroscopic forces; their form is not exact for $c \neq 0$ because of (8), which is also true in the Kirchhoff problem (see Novikov [8]).

We return to the general case and consider a domain $Q \subset M$ such that Ω is exact on $Q : \Omega = d\omega_Q$. Suppose that $x : [0, 1] \to M$ is situated entirely in Q. Then on this curve we can define the abbreviated action

$$F_Q^* = \int_0^1 (|\dot{x}(t)|_h + \omega_Q(\dot{x}(t)))\, dt,$$

where $|\cdot|_h$ is the Jacobi metric (equal to $2\sqrt{((h+L_0)L_2)}$; see §2). We fix a collection of 1-forms ω_Q for all domains Q where Ω is exact. If the curve $x(\cdot)$ lies in $Q_1 \cap Q_2$, then $\Omega = d\omega_{Q_1} = d\omega_{Q_2}$, therefore,

$$F_{Q_1}^*(x(\cdot)) - F_{Q_2}^*(x(\cdot)) = \int_{x(\cdot)} (\omega_{Q_1} - \omega_{Q_2}).$$

Since Ω is closed, by Stokes' formula the value of the integral does not change when $x(\cdot)$ varies as a curve with fixed end-points or as a closed curve. Consequently, the set of local values $F_Q^*(x(\cdot))$ determines a "many-valued functional" on the space of closed orientable curves K^+ and on the space of paths $K(x_1, x_2)$ joining $x_1, x_2 \in M$. We may assert that δF^* is a uniquely determined 1-form on K^+ (or on $K(x_1, x_2)$), however, its integral along different paths in $K^+(K)$ (which are variational curves) gives, in general, a many-valued functional on K^+ (or $K(x_1, x_2)$). Since locally F^* can be assumed to be single-valued, it has all the local properties of the classical action (in particular, Theorem 1 and the index theorem of Morse, etc. are true). The many-valued functional F^* becomes single-valued after transition to an infinite-sheeted covering $\hat{K} \to K^+$ (or $\widetilde{K}(x_1, x_2) \to K(x_1, x_2)$), however, in contrast to the classical Morse theory, the single-valued functional F^* need not be bounded below on $\hat{K}$ (or $\widetilde{K}$). This circumstance creates additional difficulties in the study of existence problems of periodic trajectories or trajectories with fixed end-points by the method of gradient descent.

Many-valued functionals were introduced by Novikov. In his papers(1) [6] - [8] an extended Morse theory is constructed for the case of periodic variational problems. Before passing to a brief account of Novikov's results we give two examples to show that Morse theory is not applicable to many-valued functionals in $K(x_1, x_2)$. The first of these supplements Example 3.

Example 14 [6]. We consider the problem of the motion of a charged particle in a constant magnetic field in $\mathbf{R}^2$ (see Example 11). For a fixed value of the energy, since the Larmor radius is bounded, it is impossible to

(1)The reader should be aware of certain inaccuracies in [6] - [8]. They are connected with the fact that in these papers the space of oriented closed curves without self-intersections is discussed. However, the application of the gradient descent in the irreversible case may lead to the appearance of self-intersections. For a somewhat more precise account, see [24].

join any two points $\mathbf{R}^2$ by an extremal of F^*. The reason is that F^* is unbounded below on $K(x_1, x_2)$. For when we join any two points $x_1, x_2 \in \mathbf{R}^2$ by a long curve γ_1 and a short one γ_2, then clearly, $F^*(\gamma_1) \sim F^*(\gamma_2^{-1}\gamma_1)$. The action F^* on the closed curve $\gamma_2^{-1}\gamma_1$ is made up of two quantities: one of them is proportional to the length of γ_1 and the other to the area inside $\gamma_2^{-1}\gamma_1$. By increasing γ_1 and choosing its orientation we can make $F^*(\gamma_1)$ tend to $-\infty$. It may seem that this phenomenon is due to the fact that $\mathbf{R}^2$ is non-compact. The next example shows that this is not the case.

Example 15 [6]. We consider the system with gyroscopic forces of Example 12. For a fixed value of the total energy E and large values of the intensity of the magnetic field H the Larmor radius is small (see (7)), this again leads to the conclusion that the two-point variational problem is unsoluble. In this example the configuration manifold S^2 is compact, however, F^* is unbounded.

In contrast to the two-point problem, the periodic problem of the variational calculus always has a trivial solution: the one-point curves $x(t) \equiv x_0$ at which F^* has a local minimum (see 1.3).

3.2. Periodic trajectories of systems with gyroscopic forces.

Theorem 7 [6]. *If the configuration space M is compact and simply-connected and if $H^2(M) \neq 0$, then for all values of the total energy $h > \max(-L_0)$ the equations of motion have a periodic solution with the given energy $h = L_2 - L_0$.*

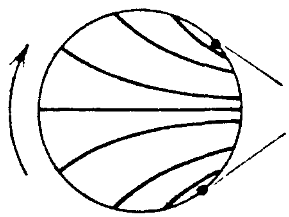

Fig. 8

The validity of Theorem 7 can be checked by the following arguments. As already mentioned, F^* always has the trivial one-point extremals $x(t) \equiv x_0$. They form an n-dimensional submanifold $N \subset K^+$ that is diffeomorphic to M. Each of these extremals is a local minimum of F^*. On any sheet of the cover $f: \hat{K} \to K^+$ the full inverse image

$$f^{-1}(N) = N_0 \cup N_1 \cup \ldots$$

gives a manifold of local minima of F^*. Since M is simply-connected, there is a natural homotopy $g: M \times [0, 1] \to \hat{K}$ joining N_0 to N_1 (Fig. 8). We restrict F^* to $M \times [0, 1]$ and begin to "shift" g downwards relative to the gradient of F^*; here the ends N_0 and N_1 are not moved. Since $H^2(M) \neq 0$, we see that $\pi_2(M) \neq 0$, consequently, K^+ is not simply-connected. The gradient descent gives us the required non-trivial stationary "saddle" critical point.

Theorem 7 has a number of non-trivial applications.

***Theorem* 8.** *The equations of the problem of the rotation of a rigid body about a fixed point in an axisymmetric field, for fixed values of the area constant $c = \langle M, e \rangle$ and energy $h >$ max V_c (here $V_c = V + c^2/(2I\langle e, e \rangle)$ is the reduced potential), have at least one periodic motion.*

The proof follows from Theorem 7, when we take into account that S^2 is compact and simply-connected, S^2 being the configuration space of the system (see Example 13). If $c = 0$, then we can assert more: for all $h >$ max V_0 the equations of rotation of a rigid body have at least six distinct periodic trajectories (their projections onto the sphere are three distinct closed non-self-intersecting curves). The fact is that for $c = 0$ the system is natural and for a fixed value of $h >$ max V_0 the Riemannian space $(S^2, |\cdot|_h)$ has three distinct closed geodesics (see §2.1). For the Kirchhoff equation a result similar to Theorem 8 holds (see [8]).

Remark. In the case of a homogeneous force field, in the dynamics of a rigid body there are many particular solutions which for the most part are periodic. Using, for example, a result of Steklov [32], we can prove that there are periodic solutions on all compatible non-critical levels of the energy and momentum integrals (and not only for sufficiently large values of h), provided that the centre of mass of the body lies on the axis of inertia. The Steklov periodic solutions can be expressed in terms of elliptic functions of time.

In the case of non-simply-connected manifolds we can use the following proposition to prove the existence of closed trajectories.

***Proposition* 8.** *If there is a closed curve γ on which $F^* < 0$ that is homotopic to zero, then for $h >$ max$(-L_0)$ there is at least one periodic trajectory.*

The proof can be deduced from the fact that F^* has a local minimum zero on one-point curves. If F^* is negative on a curve γ homotopic to zero, then an application of the method of gradient descent gives us a saddle critical point. When M is simply-connected, Theorem 7 and Proposition 8 give different periodic trajectories.

As an example we consider the motion of a charged particle in $\mathbf{R}^2 = \{x, y\}$ in a magnetic field that is periodic in x and y (see Example 11). If the mean value $\bar{H} \neq 0$, then there are closed curves in $\mathbf{R}^2$ on which $F^* < 0$ (on $\mathbf{T}^2$ they are homotopic to zero). Consequently, there is a periodic trajectory of any positive total energy. (Here the charged particle rotates in the same way as in a constant magnetic field.)

3.3. Applications of the generalized geometric theorem of Poincaré. In some cases the existence of periodic trajectories of mechanical systems with gyroscopic forces can be established by a generalization of the well-known geometric theorem of Poincaré on fixed points of symplectic diffeomorphisms. As an example we consider the motion of a charge in the "Euclidean" two-dimensional torus $\mathbf{T}^2 = \{x, y \bmod 2\pi\}$ under the action of a magnetic field with intensity $H: \mathrm{T}^2 \to \mathbf{R}^2$ (see Example 1). The motion of the charge can be described by the equations

$$\ddot{x} = -H(x, y)\dot{y}, \quad \ddot{y} = H(x, y)\dot{x}.$$

The total energy $(\dot{x}^2+\dot{y}^2)/2 = h$ is, of course, preserved.

***Theorem* 9**[1]. *If H does not vanish, then for each fixed value of $h > 0$ there are at least four closed trajectories, counting multiplicities, and at least three are geometrically distinct. If $H^2(x, y) > h$ for all $(x, y) \in \mathrm{T}^2$, then there are at least four (counting multiplicities) closed trajectories homotopic to zero.*

Proof[2]. For all $h > 0$ the energy surface $\dot{x}^2+\dot{y}^2 = 2h$ is diffeomorphic to the three-dimensional torus $\mathbf{T}^3$ with angular coordinates $x, y, \varphi = \arctan(\dot{y}/\dot{x})$. The equations of motion on $\mathbf{T}^3$ have the following form:

$$\dot{x} = \sqrt{2h}\cos\varphi, \quad \dot{y} = \sqrt{2h}\sin\varphi, \quad \dot{\varphi} = H(x, y).$$

Since $H \neq 0$, the angular variable φ varies monotonically. To be definite, suppose that $\dot{\varphi} > 0$. We rewrite the equations of motion, taking φ as a new "time":

$$x' = \frac{\sqrt{2h}\cos\varphi}{H}, \quad y' = \frac{\sqrt{2h}\sin\varphi}{H}; \quad (\cdot)' = \frac{d(\cdot)}{d\varphi}.$$

The phase flow of these equations preserves the symplectic structure $H(x, y)dx \wedge dy$. Let $x \mapsto x + f(x, y)$, $y \mapsto y + g(x, y)$ by a symplectic transformation of $\mathbf{T}^2 = \{x, y \bmod 2\pi\}$ onto itself, which is the map after time $\varphi = 2\pi$ (Fig. 9).

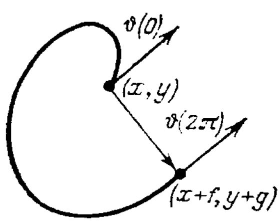

Fig. 9

[1] See Theorem 2 of [24].

[2] The idea of this proof was found independently by Arnol'd.

It can be shown that this map preserves the centre of gravity of $\mathbf{T}^2$, that is,

$$\iint_{\mathbf{T}^2} fH\, dx \wedge dy = \iint_{\mathbf{T}^2} gH\, dx \wedge dy = 0.$$

According to the generalized Poincaré theorem as stated by Arnol'd ([25], Appendix 9), and the completeness proved in [11], this map has at least four fixed points per period (counted with multiplicities), among which there are necessarily three that are geometrically distinct. To complete the proof it remains to verify that $|f|$, $|g| < 2\pi$ if min $H^2 > h$. For example,

$$|f|^2 = \left| \int_0^{2\pi} \frac{\sqrt{2h} \cos \varphi \, d\varphi}{H(x(\varphi), y(\varphi))} \right|^2 \leqslant 2h \int_0^{2\pi} \cos^2 \varphi \, d\varphi \int_0^{2\pi} \frac{d\varphi}{H^2} \leqslant \frac{(4\pi)^2 h}{\min H^2},$$

hence $|f| \leqslant 2\pi(\sqrt{h}/\min |H|)$, as required.

Remark. We complicate the problem by adding potential forces with potential $V : \mathbf{T}^2 \to \mathbf{R}$. We consider the motion of a charge under the condition that max $V < h$. Since

$$\dot{\varphi} = H + \frac{V'_y \dot{x} - V'_x \dot{y}}{2(h - V)},$$

φ varies monotonically if

$$|H| > \sqrt{V_x'^2 + V_y'^2} / \sqrt{2(h - V)}.$$

This inequality guarantees the existence of three periodic trajectories with energy h. If $h <$ max V, then φ does not vary monotonically everywhere, therefore, in this case we cannot say anything definite on the presence of closed trajectories.

We mention one approach to the four cycles on the torus with a magnetic field as suggested by Arnol'd. For this purpose, fixing the centre of gravity of a disc on the torus and its "magnetic" area

$$\iint H\, dx \wedge dy,$$

we minimize the length of the boundary. If the resulting function of the centre of gravity, regarded as a point on $\mathbf{T}^2$, turns out to be smooth, then its critical points (and there are at least four, counting multiplicities) give us closed trajectories bounding a fixed area. Varying the area between zero and infinity, we obtain closed trajectories of a given energy. This programme has not yet been realized. The attraction of this approach lies in the possibility of generalizing it to surfaces other than tori. True, we must give a suitable definition of the centre of gravity. Apparently, the number of distinct closed trajectories on any compact surface M with a non-zero magnetic field is bounded below for all $h > 0$ by the category of M.

We discuss next the problem of the motion of a charge on the sphere in a magnetic field (Example 12). In the absence of a magnetic field the point moves periodically over long curves. Seifert's theorem [2] implies the existence of periodic trajectories when a small magnetic field is added.

§4. Asymptotic solutions. Application to the theory of stability of motion

In this section we consider the motions of mechanical systems that tend to positions of equilibrium as time increases without bound. To this problem we can reduce the study of motions asymptotic to a given motion of arbitrary form (not just to equilibria). For let $x_0(\cdot)$ be a solution of Lagrange's equation $[L(\dot{x}, x, t)]_x = 0$. We put $y = x - x_0(t)$ and $\hat{L}(\dot{y}, y, t) = L(\dot{y}+\dot{x}_0, y+x_0, t)$. Obviously, $y(t) \equiv 0$ is a solution of $[\hat{L}_y] = 0$. If $x(t) \to x_0(t)$ as $t \to \infty$, then $y(t) \to 0$.

4.1. The existence of asymptotic motions.

We consider a non-autonomous Lagrangian system (M, L) with a smooth Lagrangian $L : TM \times \mathbf{R} \to \mathbf{R}$. Let $\langle , \rangle$ be a complete Riemannian metric on M.

Definition. The Lagrangian system (M, L) is called *regular* if there are positive constants c_1, c_2, c_3, and c_4 such that

1) $$c_1 \langle \dot{x}, \dot{x} \rangle - c_2 \leqslant L(\dot{x}, x, t),$$

2) $$c_3 \langle v, v\rangle \leqslant L''_{\dot{x}\dot{x}} v \cdot v \leqslant c_4 \langle v, v\rangle$$

for all $(\dot{x}, x, t) \in TM \times \mathbf{R}$ and $v \in TM$.

If the configuration space is compact, then the definition of regularity is independent of the choice of the Riemannian metric on M.

Example 16. Suppose that the Lagrangian is periodic (or almost periodic), depends on time, and has the form

$$L = \frac{1}{2} \langle \dot{x}, \dot{x}\rangle_t + \langle v(x, t), \dot{x}\rangle_t + U(x, t),$$

where $\langle , \rangle_t$ is a time-dependent Riemannian metric on M, v is a smooth vector field, and $U : M \times \mathbf{R} \to \mathbf{R}$ is a smooth function. 1) and 2) are necessarily satisfied if $\langle , \rangle_t$ is complete for all t and $\langle v, v\rangle_t$ and U are bounded above.

Throughout this section 1) and 2) are assumed to hold. These conditions guarantee the existence in the large of a smooth Hamiltonian function $H : T^*M \times \mathbf{R} \to \mathbf{R}$, dual (in the sense of the Legendre transformation) to the Lagrangian L. We introduce a smooth function $H_0 : M \times \mathbf{R} \to \mathbf{R}$ by restricting H to the set of points of $T^*M \times \mathbf{R}$ where the canonical momentum is $y = L'_{\dot{x}} = 0$.

We assume that the equation $L = 0$ has the solution $x(t) = a = \text{const}$, so that a is a position of equilibrium. Without loss of generality we may suppose that $H_0(a, t) = 0$.

Definition. A function $H_0 : M \times \mathbf{R} \to \mathbf{R}$ is called *negative definite* if for any neighbourhood D of a there is an $\varepsilon > 0$ such that $H(x, t) \leqslant -\varepsilon$ for all $x \notin D$ and $t \in \mathbf{R}$.

***Theorem* 10** [26]. *If H_0 is negative definite, then for any $x_0 \in M$ and $\tau \in \mathbf{R}$ there is a motion $x : [\tau, +\infty) \to M$ such that $x(\tau) = x_0$ and $x(t) \to a$ as $t \to +\infty$.*

To establish the existence of motions asymptotic to an equilibrium position a as $t \to -\infty$ it is sufficient to apply Theorem 7 to the Lagrangian system $(M, \hat{L})$, where $\hat{L}(\dot{x}, x, t) = L(-\dot{x}, x, -t)$. If $x(t)$ is a motion of the Lagrangian system (M, L), then $x(-t)$ is a motion of the system $(M, \hat{L})$.

Proof of Theorem 10. We may suppose that $L(0, a, t) \equiv 0$ for all t. From the definition of the Legendre transformation and the regularity condition it follows that

$$H = \sup_{\dot{x}} (y \cdot \dot{x} - L).$$

Consequently,

$$L(\dot{x}, x, t) \geqslant -H_0(x, t) \geqslant 0 = L(0, a, t). \tag{9}$$

Let $x_0 \neq a$ and $\tau \in \mathbf{R}$. We introduce the set $\Omega(x_0, \tau)$ of piecewise-continuously differentiable curves $x : [\tau, +\infty) \to M$ such that $x(\tau) = x_0$ and $x(t) \equiv a$ for all sufficiently large $t > \tau$. On $\Omega(x_0, \tau)$ there is defined the Hamiltonian action

$$F(x(\cdot)) = \int_\tau^\infty L(\dot{x}(t), x(t), t)\, dt.$$

Let d be the distance between points of a complete Riemannian space $(M, \langle,\rangle)$. For any curve $x(\cdot)$ from $\Omega(x_0, \tau)$ and times $t_2 > t_1 \geqslant \tau$ the Cauchy-Bunyakovskii inequality shows that

$$d^2(x(t_1), x(t_2)) \leqslant \left(\int_{t_1}^{t_2} |\dot{x}(t)|\, dt \right)^2 \leqslant (t_2 - t_1) \int_{t_1}^{t_2} |\dot{x}(t)|^2\, dt.$$

The regularity condition 1) implies that

$$d^2(x(t_1), x(t_2)) \leqslant \frac{t_2 - t_1}{c_1} (F(x(\cdot)) + c_2(t_2 - t_1)). \tag{10}$$

For $T > \tau$ we denote by Ω_T the set of curves $x(\cdot) \in \Omega(x_0, \tau)$ such that $x(t) \equiv a$ for $t \geqslant T$. By (10), F is uniformly bounded and uniformly continuous on any subset Ω_T on which it is bounded. Consequently, by Arzelà's theorem, when we take into account that $F \geqslant 0$, we see that $F : \Omega_T \to \mathbf{R}$ attains its greatest lower bound on some continuous curve $x_T : [\tau, T] \to M$. It follows from the regularity condition that $x_T(\cdot) \in \Omega_T$ (see [27]).

The function $T \mapsto F(x_T)$, $T > \tau$, is continuous, non-negative, and non-increasing. From (10) it follows that the family of curves $\{x_T(\cdot)\}_{T \geqslant \tau_0}$ $(\tau_0 > \tau)$ is uniformly bounded and equicontinuous. Since d is complete, by applying Arzelà's theorem again and using a diagonal process we can find a sequence $\tau_n \to +\infty$ such that for any $T > \tau$ the sequence $x_{\tau_n}(\cdot)$ converges to a continuous curve $x : [\tau, +\infty) \to M$, uniformly on $[\tau, T]$. Since $x_{\tau_n}: [\tau, T] \to M$ is a minimum of F in the class of curves with end-points at x_0 and $x_{\tau_n}(T)$, the limit curve $x : [\tau, T] \to M$ is an extremal of F in the set of curves with end-points at x_0 and $x(T) = \lim\limits_{n\to\infty} x_{\tau_n}(T)$. Consequently, $x(\cdot)$ is a motion and

$$\int_\tau^T L(\dot{x}(t), x(t), t)\, dt = \lim_{n\to\infty} \int_\tau^T L(\dot{x}_{\tau_n}(t), x_{\tau_n}(t), t)\, dt \leqslant \lim_{n\to\infty} F(x_{\tau_n}).$$

Thus,

$$(11) \qquad \int_\tau^\infty L(\dot{x}(t), x(t), t)\, dt \leqslant \inf\{F(x(\cdot)): x(\cdot) \in \Omega(x_0, \tau)\}.$$

Since H_0 is negative definite, from the convergence of (11) and from (9) it follows that $x(t) \to a$ as $t \to +\infty$.

Example 17. To the Lagrange function of Example 16 there corresponds the Hamiltonian function

$$H = \frac{1}{2}\langle y, y\rangle_t^* - y \cdot v(x, t) + \frac{1}{2}\langle v, v\rangle_t + V(x, t),$$

where $\langle , \rangle_t^*$ is the quadratic form on T_x^*M adjoint in the metric $\langle , \rangle_t$ on T_xM. If $x = a$ is a position of equilibrium, then by Theorem 10 motions asymptotic to a exist if for all $x \neq a$ and $t \in \mathbf{R}$

$$(12) \qquad \frac{1}{2}\langle v(x, t), v(x, t)\rangle_t + V(x, t) < V(a, t).$$

In the autonomous case the existence of asymptotic motions can also be established as follows. If (12) is satisfied, then the integrand for the Maupertuis action (that is, $2\sqrt{(L_0L_2)} + L_1$) is positive definite in $M\setminus\{a\}$. Consequently, the Maupertuis action attains its least value on the set of piecewise smooth curves on M with end-points at x_0 and a. This value is attained precisely on the trajectory of the reuqired asymptotic motion.

Theorem 11 [28]. *Let $H_0 : M \times \mathbf{R} \to \mathbf{R}$ be negative definite and M compact. Then there is a motion $x : \mathbf{R} \to M$ that is doubly-asymptotic to the position of equilibrium $a \in M$ (that is, $x(t) \to a$ as $t \to \pm\infty$).*

4.2. The action function in a neighbourhood of an unstable equilibrium. Again we assume that $H_0 : M \times \mathbf{R} \to \mathbf{R}$ is negative definite in a neighbourhood of a position of equilibrium $x = a$. We define a function $S : M \times \mathbf{R} \to \mathbf{R}$ by

$$S(x, \tau) = \inf\{F(z(\cdot)): z(\cdot) \in \Omega(x, \tau)\}.$$

By Theorem 10, with any point $(x, \tau) \in M \times \mathbf{R}$ we can associate an asymptotic motion $z : [\tau, +\infty) \to M$, $z(\tau) = x$, $\lim\limits_{t\to\infty} z(t) = a$.

***Proposition* 9.** *If the Lagrangian L is periodic in t then*

$$(\alpha) \qquad \lim_{t\to\infty} \dot{z}(t) = 0,$$

$$(\beta) \qquad \int_{\tau}^{\infty} L(\dot{z}(t),\ z(t),\ t)\,dt = S(x,\ \tau).$$

Proof. Let T be the period of the Lagrangian. From the convergence of (11) it follows that for large T' the Lagrangian becomes arbitrarily small at certain points of the interval $[T',\ T+T']$. But (9) shows that at these points $\dot{z}$ is small. Now (α) follows from this remark and the continuity in t of the equations of motion; (β) is a consequence of (α) and (11).

Remark. Proposition 9 is, of course, true under more general assumptions on the explicit dependence of the Lagrangian on time.

The action function S is positive definite and continuous, but need not be differentiable.

***Theorem* 12** [26]. *Let L be periodic in t, and for any time let $a \in M$ be a non-degenerate maximum of H_0. Then there is a neighbourhood $D \subset M \times \mathbf{R}$ of $\{a\} \times \mathbf{R}$ such that*

α) for any point $(x,\ \tau) \in D$ there is a unique motion $z : [\tau,\ +\infty) \to M$ asymptotic to a point a inside D;

β) S is smooth in D, has a fixed minimum on $\{a\} \times \mathbf{R}$, and satisfies the Hamilton-Jacobi equation $S'_t + H(S'_x,\ x,\ t) = 0$;

γ) if y is the momentum along the motion $z(\cdot)$, then $y(t) = S'_x(z(t),\ t)$.

By the stable manifold theorem (see, for example, [29]) the phase trajectories of the system $(M,\ L)$, asymptotic to $(y,\ x) = (0,\ a) \in T^*M$ fill out a smooth invariant submanifold $W \subset T^*M \times \mathbf{R}$, which projects diffeomorphically onto some neighbourhood of $\{a\} \times \mathbf{R}$. This proves α). We represent W as the graph of a smooth map $f : D \to T^*M \times \mathbf{R}$. If $z : [\tau,\ +\infty) \to M,\ z(\tau) = x$, is a motion asymptotic to the equilibrium $z \equiv a$, then $\dot{z}(t) = H'_y(f(z,\ t),\ z,\ t)$. By the theorem on the smooth dependence of solutions on initial data, the function $z(t,\ x,\ \tau)$, $z(\tau) = x$, is smooth and with it the action function $S(x,\ \tau)$ also depends smoothly on $(x,\ \tau) \in D$. Using Proposition 9 (β) it is easy to find that

$$dS(x,\ \tau) = y(x,\ \tau)dx - H(y(x,\ \tau),\ x,\ \tau)d\tau, \quad y = f(x,\ \tau).$$

Hence, we obtain the formulae $y = S'_x$ and $S'_\tau + H(S'_x,\ x,\ \tau) = 0$, as required.

Example 18. We consider a natural mechanical system $(M,\ \langle\,,\rangle,\ V)$. Let $a \in M$ be a non-degenerate local maximum of the potential energy V. Theorem 12 asserts that the trajectories asymptotic to a intersect the level surfaces of the action function $S(x)$ at right angles (in the sense of the metric $\langle\,,\rangle$); S itself satisfies a non-linear equation

$$\langle S'_x,\ S'_x\rangle^* = V(x) - V(a).$$

If the condition of non-degeneracy of the equilibrium does not hold, then this equation cannot have smooth solutions. Here is a simple example (see [30]):

$$S_x'^2 + S_y'^2 = x^4 + \varepsilon x^2 y^2 + y^4, \quad \varepsilon > -2. \tag{13}$$

It is easy to show that (13) for $\varepsilon \neq 2$ and $\varepsilon \neq 6$ does not have an infinitely differentiable solution in a neighbourhood of $x = y = 0$. Non-smooth solutions may exist. For example, (13) for $\varepsilon = 7$ has the solution $S(x, y) = xy\sqrt{(x^2+y^2)}$ of class C^2.

In conclusion we mention that the statement of the problem and the first results on the existence of asymptotic motions of conservative mechanical systems occur, apparently, in papers of Kneser in 1897.

4.3. A theorem on instability.

If the Lagrangian system (M, L) has a motion asymptotic to a position of equilibrium $a \in M$, then for the system $(M, \hat{L})$, where $\hat{L}$ is obtained from L by a transformation of time, this equilibrium, obviously, is unstable. Thus, according to Theorem 7, a sufficient condition for instability is that the function $H_0 : M \times \mathbf{R} \to \mathbf{R}$ is negative definite. This condition can be relaxed.

Theorem 13 [26]. *If* $H_0 \leqslant 0$ *for all* $(x, t) \in M \times \mathbf{R}$, *then for any* $\varepsilon > 0$, $x_0 \in M$, *and* $\tau_0 \in \mathbf{R}$ *there is a* $\tau > \tau_0$ *and a motion* $x : [\tau_0, \tau] \to M$ *such that* $x(\tau_0) = x_0$, $x(\tau) = a$, *and* $|\dot{x}(\tau)| \leqslant \varepsilon$.

To prove instability it is sufficient to apply the theorem to the Lagrangian system $(M, \hat{L})$. Theorem 13 is proved by the method of §4.1.

Example 19. A condition for the instability of an equilibrium for the seminatural system of Example 16 is (12) with a non-strict inequality sign. In the autonomous case this was noted by Hagedorn [31].

4.4. A compound pendulum with an oscillating point of suspension.

We apply the general results established above to the motion of a planar n-linked pendulum (see Example 6) with a vertically oscillating point of suspension. The Lagrange function has the following form:

$$L(\dot{\vartheta}, \vartheta, t) = \frac{1}{2} \sum_{i,j=1}^{n} M_{\max(i,j)} l_i l_j \cos(\vartheta^i - \vartheta^j)\, \dot{\vartheta}^i \dot{\vartheta}^j + \dot{f}(t) \sum_{i=1}^{n} M_i l_i \sin \vartheta^i \dot{\vartheta}^i - g \sum_{i=1}^{n} M_i l_i \cos \vartheta^i,$$

where $M_i = \sum_{j=i}^{n} m_j$ and $f(t)$ is the height of the point of suspension. Since the configuration space $\mathbf{T}^n = \{\vartheta \bmod 2\pi\}$ is compact, the system $(\mathbf{T}^n, L)$ is regular if $\dot{f}^2(t)$ is a smooth bounded function of time. Let $a = (\pi, \ldots, \pi)$ be the upper position of equilibrium of the pendulum.

Proposition 10. *If for all* t

$$\dot{f}^2 < g \min_s (m_s l_s / M_s), \tag{14}$$

then (12) *is satisfied.*

This remains true if in (14) and (12) the symbol $<$ is replaced by $\leqslant$.

Corollary. *For $n = 1$ the upper position of equilibrium is unstable if $\dot{f}^2(t) \leqslant gl$ for all $t \in \mathbf{R}$.*

Remark. A sufficient condition for stability in the linear approximation, obtained for $f \gg l$ by the method of averaging, has the following form: $\overline{\dot{f}^2} > gl$ (Bogolyubov [33]).

If (14) is satisfied, then by Theorem 10 there are motions of the pendulum starting at an arbitrary moment of time and in an arbitrary position and asymptotic to the upper position of equilibrium $\vartheta = a$. Moreover, according to Theorem 9, in this case there are motions of the pendulum doubly asymptotic to $a \in \mathbf{T}^n$.

Proposition 11. *If $f(t)$ is even and satisfies* (12), *then there are at least $2^n - 1$ distinct motions of the pendulum that are doubly-asymptotic to the upper position of equilibrium.*

For apart from the upper, there are $2^n - 1$ positions of equilibrium a_i that are invariant under the map $\vartheta \to -\vartheta$ (see §2.4). Let $x : [0, +\infty) \to \mathbf{T}^n$, $x(0) = a_i$, be a motion of the pendulum asymptotic to a as $t \to +\infty$. Since the map $(\dot{\vartheta}, \vartheta, t) \to (\dot{\vartheta}, -\vartheta, -t)$ preserves the Lagrangian, $\hat{\vartheta}(t) = -\vartheta(-t)$ is a motion asymptotic to a as $t \to -\infty$. Since $\dot{\hat{\vartheta}}(0) = \dot{\vartheta}(0)$, we see that $\vartheta: \mathbf{R} \to \mathbf{T}^n$ is the required doubly-asymptotic motion.

4.5. A theorem of Gaidukov.

We consider the question of the existence of geodesics on Riemannian manifolds asymptotic to closed geodesic lines. By elementary methods of the calculus of variations Gaidukov has proved the following theorem.

Theorem 14 [40]. *Let M be a smooth compact oriented two-dimensional Riemannian manifold. For any $x \in M$ and any non-trivial class π of paths in M freely homotopic to zero there is a geodesic containing x and asymptotically approximating some closed geodesic of the class π.*

By the Maupertuis principle this result holds for trajectories of a natural system with configuration space M for sufficiently large values of the total energy. With the help of Theorem 14 we can prove the following result.

Proposition 12 [41]. *Let M be a torus with angular coordinates φ_1 and φ_2. For any real λ and any (φ_1, φ_2) there is a geodesic $\varphi_1 = \varphi_1(s)$, $\varphi_2 = \varphi_2(s)$ such that*

1) $$(\varphi_1(0), \varphi_2(0)) = (\varphi_1, \varphi_2)_0,$$

2) $$\lim_{s\to\infty} \varphi_1(s)/\varphi_2(s) = \lambda.$$

In the case of a rational "rotation number" λ this result is a direct consequence of Theorem 14. We mention that the geodesics in question are minimal in the covering surface. Proposition 12 has the following more precise version.

***Proposition* 13.** *Suppose that* $\mathbf{T}^2$ *is equipped with an analytic metric, that does not differ much from the Euclidean metric, and that* λ *is a real irrational number with "good" arithmetic properties*: $|n\lambda - m| \geqslant \alpha/|n|^{\beta}$ *for some* α, $\beta > 0$. *Then there is a fibration of* $\mathbf{T}^2$ *into minimal geodesics (in the covering surface) with rotation number* λ.

It is easy to construct counterexamples for metrics on $\mathbf{T}^2$ far removed from the Euclidean metric. In addition to arguments of the calculus of variations, the proof of Proposition 13 uses results of KAM-theory. J. Moser has proved a more general theorem on the existence of a fibration into minimal hypersurfaces of a many-dimensional torus with a metric close to the standard one, moreover, the minimal surfaces in the covering space are close to hypersurfaces with "good" arithmetic properties(1).

In conclusion we state some unsovled problems.

1) To prove Theorem 14 in the case of a many-dimensional Riemannian manifold.

2) Proposition 12 apparently admits the following generalization: *if M is not homeomorphic to a sphere, then through each point in the covering space there passes a minimal geodesic asymptotically approaching a given point of the absolute.*

3) Is the assertion of 2) valid in the many-dimensional case? In particular, let M be an n-dimensional torus with angular coordinates $\varphi_1, \ldots, \varphi_n$. To prove that for any real numbers $\lambda_1, \ldots, \lambda_{n-1}$ and any $(\varphi_1, \ldots, \varphi_n)_0$ there is a geodesic $\varphi_i = \varphi_i(s)$, $1 \leqslant s \leqslant n$, such that $\varphi_i(0) = (\varphi_i)_0$ and

$$\lim_{s\to\infty} \varphi_i(s)/\varphi_n(s) = \lambda_i \qquad (1 \leqslant i \leqslant n-1).$$

4) We assume that the potential has a unique maximum in M. To prove the existence of trajectories asymptotic to the maximum point on the one side and to a closed trajectory of non-trivial homotopy class on the other side.

5) Under the same assumptions, to prove the existence of trajectories doubly-asymptotic to the maximum point and rotating around a loop of non-trivial homotopy class.

Using the method of [17] it can be proved that the number of distinct doubly-asymptotic trajectories is bounded below by the rank of the fundamental group of the configuration space. For simply-connected manifolds the existence of doubly-asymptotic trajectories was established by Bolotin [21].

4) and 5) have also been stated by Arnol'd in connection with the analysis of the transition of a Hamiltonian system through resonances (see [38]). From the point of view of this problem not only Riemannian but also "Lorentzian" metrics are of interest.

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