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ON THE INTEGRABILITY OF HAMILTONIAN SYSTEMS WITH TORAL POSITION SPACE

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ABSTRACT. This paper considers the problem on the complete integrability of a Hamiltonian system with a toral position space, with Euclidean kinetic energy and a small analytic potential. Necessary integrability conditions are found in the case when the potential is a trigonometric polynomial. These conditions are also necessary conditions of existence of additional first integrals, polynomial in the momenta (with no assumption on the smallness of the potential). The proofs are based on a detailed analysis of the classical scheme of perturbation theory. The general results are applied to the study of the complete integrability of the well-known problem on the motion of n points along a line with periodic interaction potential. In particular, the nonintegrability of the "open" chain of interactions of particles is proved for $n > 2$; the "periodic" chain is nonintegrable with the additional condition that the potential be a nonconstant trigonometric polynomial. Conditions for complete integrability of the generalized nonperiodic Toda chain are discussed.

Bibliography: 17 titles.

Introduction. Main results

Following Poincaré ([1], Chapter I, §13), we consider the "general problem of dynamics", connected with the study of Hamiltonian systems of the form

$$\dot{x}_s = -\frac{\partial H}{\partial y_s}, \quad \dot{y}_s = \frac{\partial H}{\partial x_s}, \qquad 1 \le s \le n, \ H = H_0(x) + \varepsilon H_1(x, y) + \cdots, \tag{0.1}$$

the functions $H_k(x, y)$ are assumed analytic and 2π-periodic with respect to $y_1, \ldots, y_n$; ε is a small parameter. Equations (0.1) are frequently encountered in applications.

For $\varepsilon = 0$ we have a completely integrable Hamiltonian system for which the variables x and y mod 2π are action-angle variables. Since system (0.1) has the first energy integral $\sum_{k\ge 0} H_k \varepsilon^k$, it is natural to consider the problem on the existence of additional integrals in the form of series $\sum_{k\ge 0} F_k(x, y)\varepsilon^k$ with analytic coefficients 2π-periodic with respect to y. The formulation of the problem as well as the first results in this direction belong to Poincaré (see [1], Chapter V, and [2]). With respect to generalizations, see [3] and [4]. It is well known that the problem on the existence of a complete set of independent integrals of the form $\sum F_k \varepsilon^k$ is closely related to the possibility of carrying out the classical scheme of perturbation theory (see [1], Chapter IX, [4], and [5], Chapter 4).

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In this paper we consider Hamilton's equations (0.1) whose Hamiltonians have the form

$$H = H_0(x) + \varepsilon H_1(y), \tag{0.2}$$

where $H_0 = \frac{1}{2}\sum a_{ij}x_ix_j$ is a nondegenerate quadratic form with constant coefficients, and H_1 is a trigonometric polynomial in $y_1, \ldots, y_n$. The systems with Hamiltonians (0.2) preserve the essential features of the general case; however, their analysis is technically simpler. Since the Fourier series of the perturbing function H_1 contains only finitely many harmonics, to systems with Hamiltonians (0.2) one cannot apply Poincaré's results and their well-known generalizations.

The Hamilton equations (0.1) with Hamiltonian (0.2) may be regarded as equations of motion of a mechanical system with a configuration space $\mathbf{T}^n$, a kinetic energy H_0 and a small potential εH_1. We emphasize that we do not require that the quadratic form H_0 be positive definite.

Let us agree on some notation. Let $\xi = (\xi_1, \ldots, \xi_n)$ and $\eta = (\eta_1, \ldots, \eta_n)$. We set

$$(\xi, \eta) = \sum_{i=1}^{n} \xi_i\eta_i, \qquad \langle \xi, \eta \rangle = \sum_{i,j=1}^{n} a_{ij}\xi_i\eta_j.$$

Let

$$H_1 = \sum h_m e^{i(m,y)}, \qquad h_m = \text{const}. \tag{0.3}$$

We denote by $\mathfrak{M}$ the finite set of integer vectors $m = (m_1, \ldots, m_n)$ such that $h_m \neq 0$. If $H_1 \neq \text{const}$, then $\mathfrak{M}$ contains at least two elements. We remark also that $\mathfrak{M}$ admits the involution $m \mapsto -m$.

DEFINITION 1. A Hamiltonian system with the Hamiltonian $H_0 + \varepsilon H_1$ is called *Poincaré integrable* if there exist n integrals in the form of power series

$$\begin{aligned} F^{(1)} &= F_0^{(1)}(x, y) + \varepsilon F_1^{(1)}(x, y) + \cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F^{(n)} &= F_0^{(n)}(x, y) + \varepsilon F_1^{(n)}(x, y) + \cdots, \end{aligned} \tag{0.4}$$

whose coefficients are analytic functions on $\mathbf{R}^n \times \mathbf{T}^n$, and the functions $F_0^{(1)}, \ldots, F_0^{(n)}$ are independent almost everywhere.

In connection with this definition we make several remarks.

1) We do not assume that the series (0.4) are convergent for small $\varepsilon \neq 0$. This requires some explanation. The formal series $\sum f_s\varepsilon^s$ is assumed to be zero if all $f_s \equiv 0$. The series $F = \sum F_k\varepsilon^k$ is a formal integral of the Hamilton equations with Hamiltonian $H = \sum H_m\varepsilon^m$ if the formal series

$$\{H, F\} = \sum_{s\geq 0} \left(\sum_{m+k=s} \{H_m, F_k\} \right) \varepsilon^s$$

equals zero; here $\{\ ,\ \}$ denotes the standard Poisson bracket.

2) We do not assume that the integrals (0.4) are involutive. It turns out that, since H_0 is nondegenerate, any two integrals of (0.1) are automatically in involution (cf. [6], §5).

3) In the case of two degrees of freedom ($n = 2$) the condition of independence of the functions H_0 and F_0 may be replaced by the weaker and more natural condition of nonidentical dependence (with respect to ε) of the integrals $H_0 + \varepsilon H_1$ and $\sum F_s\varepsilon^s$ (see [1], Chapter V). More precisely, several formal series are assumed to be independent

if some minor of maximal order of their Jacobi matrix is different from zero, the minor being regarded as a formal series in ε.

Let us now set $\varepsilon = 1$ and consider a Hamiltonian system with Hamiltonian function $H_0 + H_1$.

DEFINITION 2. This Hamiltonian system is called *Birkhoff integrable* if there exists a complete set (n altogether) of polynomial integrals with respect to the momenta $x_1, \ldots, x_n$ with analytic and 2π-periodic coefficients with respect to $y_1, \ldots, y_n$, and their highest-order homogeneous forms are independent almost everywhere.

We make some more remarks.

1) In the case of two degrees of freedom we replace the condition of independence of the highest-order forms of the polynomial integrals by the weaker condition of independence of the integrals as analytic functions in $\mathbf{R}^2 \times \mathbf{T}^2$.

2) We do not know of examples of completely integrable natural Hamiltonian systems having no complete set of polynomial integrals.

3) It is well known (see, for instance, [7] and [8]) that integrals linear in the momenta are connected with the existence of "hidden" cyclic coordinates, and integrals that are quadratic in the momenta are connected with the existence of separated canonical variables. As Birkhoff showed, these conclusions are valid also for polynomial functions that are integrals at a certain level of the energy integral of the system with two degrees of freedom (see [8], Chapter II).

PROPOSITION 1. *If a system with Hamiltonian $H_0 + H_1$ is Birkhoff integrable, then a system with Hamiltonian $H_0 + \varepsilon H_1$ is Poincaré integrable.*

For the proof we use the change of variables

$$y \mapsto y, \quad x \mapsto x/\sqrt{\varepsilon}, \quad t \mapsto \sqrt{\varepsilon}t. \tag{0.5}$$

After that equations (0.1) are transformed into Hamilton's equations with Hamiltonian $H_0 + \varepsilon H_1$, and the polynomial integral becomes $F + \sqrt{\varepsilon}\Phi$ (up to an unessential constant factor), where F and Φ are functions analytic with respect to ε. Clearly, F and Φ are integrals of the system with Hamiltonian $H_0 + \varepsilon H_1$, and one of the independent terms F_0 or Φ_0 coincides with the homogeneous form of highest degree of the initial polynomial integral, as required.

As S. V. Bolotin has observed, the converse is also true. Indeed, let us assume that the series

$$\sum F_s(x, y)\varepsilon^s, \qquad F_s: \mathbf{R}^n \times \mathbf{T}^n \to \mathbf{R}, \tag{0.6}$$

is an integral of a Hamiltonian system with Hamiltonian function $H_0 + \varepsilon H_1$. The inverse change of variables with respect to (0.5) takes this system into one with Hamiltonian $H_0 + H_1$. The integral (0.6) is transformed into

$$\sum F_s(\sqrt{\varepsilon}x, y)\varepsilon^s = \sum \Phi_m(x, y)(\sqrt{\varepsilon})^m,$$

where the Φ_m are polynomials in the momenta with periodic coefficients with respect to y. Since this system does not contain the parameter ε, the polynomials Φ_m are its integrals. In the case of two degrees of freedom this immediately implies the existence of a polynomial integral independent of the function $H_0 + H_1$. The problem on the existence of integrals with independent forms of highest degree in the multidimensional case requires further study. In what follows we use only Proposition 1. We remark that if the perturbing function H_1 depends on the momenta x, Proposition 1 is no longer valid.

The main result of our paper is the following.

THEOREM 1. *Let the quadratic form H_0 be positive definite. Then a Hamiltonian system with Hamiltonian function $H_0 + \varepsilon H_1$ is Poincaré (Birkhoff) integrable if and only if the points of the set $\mathfrak{M}$ lie on $d \leq n$ lines that intersect orthogonally (in the metric $\langle\ ,\ \rangle$) at the origin.*

The proof of sufficiency is very simple. In fact, let $l_1, \ldots, l_d$ be the lines in $\mathbf{R}^n$ referred to in Theorem 1. We denote by $k_i \neq 0$ the nearest point of the set $\mathbf{Z}^n \cap l_i$ to the origin, and complement $k_1, \ldots, k_d$ by integer vectors $k_{d+1}, \ldots, k_n$ to a basis in $\mathbf{R}^n$. Then we make a linear transformation $y' = My$ with the nondegenerate integer matrix

$$M = \begin{Vmatrix} k_1 \\ \vdots \\ k_n \end{Vmatrix}$$

and extend it to a canonical transformation $x, y \mapsto x', y'$ by setting $x' = (M^T)^{-1}x$. In the new variables x', y' the Hamiltonian function $H_0 + H_1$ is reduced to

$$H = \frac{1}{2}\left(\sum_{i=1}^{d} a'_{ii}x_i'^2 + \sum_{j=1}^{n}\sum_{i>d} a'_{ij}x'_ix'_j\right) + \sum_{i=1}^{d} f_i(y'_i), \tag{0.7}$$

where $a'_{ij} = \text{const}$, and the f_i are 2π-periodic trigonometric polynomials. Clearly the variables x', y' are separated and hence Hamilton's equations with Hamiltonian (0.7) have the following set of n independent involutive integrals:

$$F_i = \frac{1}{2}\left(a'_{ii}x_i'^2 + x'_i\sum_{s>d} a_{is}x'_s\right) + f_i(y'_i), \qquad 1 \leq i \leq d, \qquad F_j = x'_j, \qquad j > d.$$

In the original variables x, y the integrals F_i $(i \leq d)$ are again quadratic in the impulse functions with analytic coefficients on $\mathbf{T}^n = \{y\}$, and the integrals F_j $(j > d)$ remain linear functions of x with constant coefficients.

COROLLARY 1. *If Hamilton's equations with Hamiltonian $H_0 + H_1$ have n polynomial integrals with independent forms of highest degree, then they have n independent involutive polynomial integrals of degree no higher than two.*

Consider k orthogonal lines in $(\mathbf{R}^n, \langle\ ,\ \rangle)$ intersecting at the origin, and take two points on each line, lying at the same distance but in opposite directions from the point $0 \in \mathbf{R}^n$. We call the convex hull of these $2k$ points a *k-dimensional rhomboid*. The number of l-dimensional faces of a k-dimensional rhomboid equals $2^{l+1}\binom{k}{l+1}$; in particular, this polyhedron has precisely $2k$ vertices and 2^k faces. Clearly, a k-dimensional rhomboid is a convex polyhedron dual to a k-dimensional parallelepiped.

COROLLARY 2. *If Hamilton's equations with Hamiltonian $H_0 + \varepsilon H_1$ are Poincaré (Birkhoff) integrable, then the convex hull $\mathscr{C}(\mathfrak{M})$ is a k-dimensional rhomboid, $k \geq n$.*

As an example let us consider a system with a potential of the form

$$H_1 = \sum_{i<j} f(y_i - y_j), \tag{0.8}$$

where $f(\cdot)$ is an even function that is a nonconstant 2π-periodic trigonometric polynomial (potential of twin interaction). One can prove that in this problem the convex hull of the set $\mathfrak{M}$ is an $(n-1)$-dimensional polyhedron with $2\binom{n}{2}$ vertices. Since

$2\binom{n}{2} > 2(n-1)$ for $n > 2$, then for $n \geq 3$ a system with potential (0.8) has no complete set of polynomial integrals. This conclusion does not depend on the form of the Euclidean metric $\langle\ ,\ \rangle$.

A particular case of this problem is considered in the paper of Adler and van Moerbeke [9]: $\langle\ ,\ \rangle$ is the standard metric in $\mathbf{R}^n$, $f = \cos(\cdot)$. This is a classical alternate version of the Gross-Neveu system, which is well known in theoretical physics. Using Kovalevskaya's Ansatz, it is proved that for $n = 3$ and $n = 4$ for almost all initial conditions the variables x_s and $\exp(iy_s)$ are not meromorphic functions of complex time. In particular, the Gross-Neveu system is not integrable. We emphasize the fact that algebraically nonintegrable systems may be completely integrable. As a simple example let us consider a Hamiltonian system with one degree of freedom, whose Hamiltonian function equals

$$x^2/2 + f_n(y), \tag{0.9}$$

where f_n is a polynomial of degree n with simple roots. This system is algebraically integrable only for $n \leq 4$; however, it is completely integrable for all n in the real domain due to the existence of the polynomial integral (0.9).

It is interesting to observe that the system with Hamiltonian

$$\frac{1}{2}\sum x_s^2 + \sum_{i<j} f(y_i - y_j), \tag{0.10}$$

where f is the Weierstrass $\wp$-function (or its degenerate cases z^{-2}, $\sin^{-2}(z)$, and $\sinh^{-2}(z)$) is completely integrable for all values of n (see [10] and [11]). In [12] it is proved that in the case of three particles this is the only case when a Hamiltonian system with Hamiltonian function (0.10) admits an additional integral in the form of a polynomial of third degree in the momenta. We note that the problem on the existence of an additional polynomial integral of a given degree is much simpler than that of the existence of an integral in the form of a polynomial whose degree is not specified beforehand.

Let us introduce in $\mathbf{Z}^n$ the standard relation of lexicographic order, which we denote in the sequel by $\prec$: $\sigma \prec \delta$, if for the least index s such that $\sigma_s \neq \delta_s$ we have $\sigma_s < \delta_s$. We shall say that $\sigma \preceq \delta$ if either $\sigma \prec \delta$ or $\sigma = \delta$.

DEFINITION. Let α be the maximal element of $\mathfrak{M}$, and let β be the maximal element of the set $\mathfrak{M}\backslash\{\alpha\}$ linearly independent with α. We shall call the vector α the *vertex of* $\mathfrak{M}$, and β the *vertex of* $\mathfrak{M}$ *adjoining* α.

Leaving aside the trivial integrability case when all points of $\mathfrak{M}$ lie on one line passing through the origin, we assume in the sequel that the adjoining vertex β always exists.

The proof of Theorem 1 relies upon the application of the following assertion, which is of independent interest.

THEOREM 2. *Let α and β be vertices of $\mathfrak{M}$, and assume that*

$$m\langle\alpha,\alpha\rangle + 2\langle\alpha,\beta\rangle \neq 0 \tag{0.11}$$

for every integer $m \geq 0$. Then a Hamiltonian system with Hamiltonian function $H_0 + \varepsilon H_1$ is not Poincaré integrable.

We emphasize that for the validity of Theorem 2 one needs only the nondegeneracy of the quadratic form H_0. Theorem 2 is proved using perturbation theory. It turns out that the independent coefficients of the integrals (0.4)—the functions $F_0^{(s)}$—do not contain angle coordinates y and are dependent at all points of the hyperplanes

$\langle x, m\alpha + \beta\rangle = 0$. By the analyticity and the assumption on the linear independence of α and β, the functions $F_0^{(s)}$ are dependent everywhere on $\mathbf{R}^n = \{x\}$. The points $x \in \mathbf{R}^n$ lying on the hyperplane $\langle x, m\alpha + \beta\rangle = 0$ correspond to resonance tori of the nonperturbed integrable problem which collapse on the mth step of the perturbation theory.

THEOREM 3. *Let α and β be vectors in $\mathfrak{M}$ satisfying the assumptions of Theorem 2. If the Hamiltonian system with Hamiltonian function $H_0 + \varepsilon H_1$ has $n-1$ single-valued analytic integrals*

$$F_0^{(1)} + \varepsilon F_1^{(1)} + \cdots, \ldots, F_0^{(n-1)} + \varepsilon F_1^{(n-1)} + \cdots,$$

and the functions $F_0^{(1)}, \ldots, F_0^{(n-1)}$ are independent at least at one point of $\Gamma \times \mathbf{T}^n$, where Γ is the hyperplane $\langle \alpha, x\rangle = 0$, then Hamilton's equations do not have an integral independent of the functions $F_0^{(1)} + \varepsilon F_1^{(1)} + \cdots$ and $F_0^{(n-1)} + \varepsilon F_1^{(n-1)} + \cdots$ in the form of a formal series $\sum F_s(x, y)\varepsilon^s$ with analytic coefficients in $\mathbf{R}^n \times \mathbf{T}^n$.

Theorem 3 and Proposition 1 imply the following.

COROLLARY. *Assume that the vectors $\alpha, \beta \in \mathfrak{M}$ satisfy the assumptions of Theorem 2, and let Γ be the hyperplane $\langle \alpha, x\rangle = 0$. Assume that the system with Hamiltonian $H_0 + H_1$ has $n-1$ polynomial integrals $F^{(1)}, \ldots, F^{(n-1)}$ whose homogeneous forms of highest degree are independent at least at one point of $\Gamma \times \mathbf{T}^n \subset \mathbf{R}^n \times \mathbf{T}^n$. Then Hamilton's equations do not have an additional polynomial integral independent of the functions $F^{(1)}, \ldots, F^{(n-1)}$.*

The plan of the subsequent exposition is as follows. §1 contains auxiliary results connected with the detailed analysis of the classical scheme of perturbation theory as applied to a Hamiltonian system with Hamiltonian function (0.2). Using these results, in §2 we prove Theorems 2 and 3. §3 contains the deduction of Theorem 1 from Theorem 2. The paper is completed with the discussions in §4 of the possible ways of generalizing Theorems 2 and 3, and some applications.

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§1. The secular set and its structure

We begin the proof of Theorems 2 and 3 with the exposition of some notions of perturbation theory. The classical scheme of perturbation theory consists in the following: one looks for a canonical transformation $x, y \bmod 2\pi \mapsto u, v \bmod 2\pi$:

$$x_i = \partial S/\partial y_i, \quad v_i = \partial S/\partial u_i, \qquad i = 1, \ldots, n,$$

with generating function

$$S = S_0(u, y) + \varepsilon S_1(u, y) + \cdots,$$

taking the initial Hamiltonian $H_0(x) + \varepsilon H_1(y)$ into a function $K_0(u) + \varepsilon K_1(u) + \cdots$ independent of the new angle variables v. If one succeeds in finding such a transformation, the original system is integrated. In particular, the functions $u_1(x, y, \varepsilon), \ldots, u_n(x, y, \varepsilon)$ constitute a complete set of independent Newtonian integrals. The generating function S satisfies the Hamilton-Jacobi equation

$$H_0(\partial S/\partial y) + \varepsilon H_1(y) = K_0(u) + \varepsilon K_1(u) + \cdots. \tag{1.1}$$

One usually sets $S_0 = (u, y)$; then for $\varepsilon = 0$ one has the identity map and therefore $K_0(u) = H_0(u)$. Expanding the left-hand side of (1.1) in series in powers of ε and

equating the coefficients of equal powers of ε, we obtain an infinite chain of equations for the successive determination of $S_1, S_2, \ldots$ and $K_1, K_2, \ldots$:

$$\begin{aligned} \sum_l \frac{\partial H_0}{\partial u_l}\frac{\partial S_1}{\partial y_l} + H_1(y) &= K_1(u), \\ \sum_l \frac{\partial H_0}{\partial u_l}\frac{\partial S_m}{\partial y_l} + \frac{1}{2}\sum_{i,j} a_{ij}\sum_{p,q}\frac{\partial S_p}{\partial y_i}\frac{\partial S_q}{\partial y_j} &= K_m(u). \end{aligned} \tag{1.2}$$

In the deduction of these formulas we have taken into account the fact that H_0 is a quadratic form in u. In perturbation theory one proves (see, for instance, [1], Chapter IX) that equations (1.2) have a unique solution $S_1, S_2, \ldots$ in the form of trigonometric series in $y_1, \ldots, y_n$ with zero independent coefficients:

$$S_m = \sum_{\tau \in \mathbf{Z}^n}{}' S_m^\tau(x) e^{i(\tau, y)}. \tag{1.3}$$

Let us consider the first equation of (1.2) and solve it by the Fourier method. Using (0.3) and (1.3) (for $m = 1$), we obtain

$$K_1 = h_0, \quad S_1^\tau = \frac{ih_\tau}{(\omega, \tau)}, \qquad \tau \neq 0; \tag{1.4}$$

here $\omega = (\omega_1, \ldots, \omega_n)$, where the $\omega_s = \partial H_0/\partial x_s = \sum a_{is}x_i$ are the frequencies of the quasiperiodic motions of the unperturbed problem. From (1.4) one sees that S_1 is not determined at the points of $\mathbf{R}^n = \{x\}$ lying on finitely many hyperplanes $\langle x, \tau\rangle = 0$, $\tau \in \mathfrak{M}$, $\tau \neq 0$. We call the collection of all these points the *first order secular set*, and denote it by $\mathbf{B}_1$.

The Fourier coefficients S_m^τ, $m = 2, 3, \ldots$, are found by the inductive formula

$$S_m^\tau = \frac{1}{2i(\omega, \tau)} \sum_{\substack{u+v=m \\ \sigma+\delta=\tau}} \langle \sigma, \delta\rangle S_u^\sigma S_v^\delta. \tag{1.5}$$

This is a consequence of (1.2) and the notation (1.3). Clearly, the S_r^τ may be represented as fractions whose denominators contain expressions of the form (ω, τ) and their products.

By the *kth order secular set* $\mathbf{B}_k$ we mean the set of all points of $\mathbf{R}^n = \{x\}$ such that the following conditions hold:

i) $(\omega(x), \tau) = 0$, $\tau \neq 0$.

ii) $(\omega(x), \tau)S_k^\tau(x) \not\equiv 0$.

iii) On the hyperplane $(\omega, \tau) = 0$ all functions S_m^σ are analytic for $m < k$.

We set $\mathbf{B} = \bigcup_1^\infty \mathbf{B}_k$. We call this set the *secular set* of the perturbed problem. Since the points of $\mathbf{B}$ are points of discontinuity for the Fourier coefficients of the function S, in the sequel its structure plays an important role. Clearly, each set $\mathbf{B}_1 \bigcup \cdots \bigcup \mathbf{B}_k$ consists of finitely many different hyperplanes.

MAIN LEMMA. *Assume that the vertices α and β of $\mathfrak{M}$ satisfy* (0.11). *Then the set $\mathbf{B}_k$ contains the hyperplane $\langle k\alpha + \beta, x\rangle = 0$. In particular, the secular set $\mathbf{B}$ consists of infinitely many different hyperplanes, and its closure contains the hyperplane $\langle \alpha, x\rangle = 0$.*

The rest of §1 is devoted to the proof of the Main Lemma.

From the definition of the lexicographic order it follows that $\alpha \succ 0$ and $\alpha \succ \gamma$ for all $\gamma \in \mathfrak{M}$.

LEMMA 1. $S_r^\tau \equiv 0$ *for all* $\tau \succ r\alpha$.

The proof is by induction over r. For $r = 1$ the lemma follows from (1.4) and the definition of the vertex α. Assume that the lemma holds for all $r \leq m$. The function S_{r+1}^τ is computed by (1.5). Let $\tau \succ (r+1)\alpha$. Let us prove that any term in the right-hand side of (1.5) must contain the factor S_w^τ for $\tau \succ w\alpha$ and $w \leq r$, which equals zero by the inductive hypothesis. Indeed, if $\sigma \preceq u\alpha$ and $\delta \preceq v\alpha$, then $\sigma + \delta \preceq (u+v)\alpha = (r+1)\alpha \prec \tau$. But this contradicts the summation condition $\sigma + \delta = \tau$. The lemma is proved.

LEMMA 2.

$$S_m^{m\alpha} = \frac{\langle \alpha, \alpha \rangle}{2im(\omega, \alpha)} \sum_{u+v=m} uv S_u^{u\alpha} S_v^{u\alpha}. \tag{1.6}$$

PROOF. We deduce (1.6) from (1.5), setting $\tau = m\alpha$. We consider only the nonzero terms in the right-hand side. According to Lemma 1 we have $\sigma \preceq u\alpha$, $\delta \preceq v\alpha$, and $\sigma + \delta = m\alpha = (u+v)\alpha$. Hence $\sigma = u\alpha$ and $\delta = v\alpha$, as required.

LEMMA 3.

$$S_m^{m\alpha} = K_m \left(\frac{\langle \alpha, \alpha \rangle}{i(\omega, \alpha)} \right)^{m-1} (S_1^\alpha)^m, \tag{1.7}$$

where

$$K_1 = 1, \qquad K_m = \sum_{u+v=m} \frac{uvK_uK_v}{2m}. \tag{1.8}$$

The proof is by induction over m. For $m = 1$, (1.7) coincides with (1.4). Assume that Lemma 3 holds for $m \leq r$. Then

$$S_{r+1}^{(r+1)\alpha} = \frac{\langle \alpha, \alpha \rangle \sum_{u+v=r+1} uvK_uK_v}{2i(r+1)(\omega, \alpha)} \left(\frac{\langle \alpha, \alpha \rangle}{i(\omega, \alpha)} \right)^{u+v-2} (S_1^\alpha)^{u+v}$$
$$= K_{r+1} \left(\frac{\langle \alpha, \alpha \rangle}{i(\omega, \alpha)} \right)^r (S_1^\alpha)^{r+1}.$$

LEMMA 4. *If the vector* τ *is linearly independent of* α *and* $(m-1)\alpha + \beta \prec \tau \prec m\alpha$, *then* $S_m^\tau \equiv 0$.

The validity of this assertion for $m = 1$ follows from the definition of the vertices α and β. Assume that it holds for all $m < r$. We use (1.5) for $m = r+1$. By the inductive hypothesis and Lemma 1, the product $S_u^\sigma S_v^\delta$ can be nonzero only in the following cases:

1) the vectors α, δ, and σ are pairwise linearly independent; or

2) either $\sigma \preceq u\alpha$ and $\delta \preceq (v-1)\alpha + \beta$, or $\sigma \preceq (u-1)\alpha + \beta$ and $\delta \preceq v\alpha$.

In the first case the vector τ is clearly parallel to α, and in the second we have $\tau = \sigma + \delta \preceq (u+v-1)\alpha + \beta = r\alpha + \beta$, as required.

LEMMA 5.

$$S_{m+1}^{m\alpha+\beta} = \frac{1}{i(\omega, m\alpha + \beta)} \sum_{\substack{u+v=m \\ u>0, v\geq 0}} \langle u\alpha, v\alpha + \beta \rangle S_u^{u\alpha} S_{v+1}^{v\alpha+\beta}. \tag{1.9}$$

Formula (1.9) follows from (1.5) and Lemma 4. We first note that either $\sigma \preceq (u-1)\alpha + \beta$ or $\delta \preceq (v-1)\alpha + \beta$. Otherwise the vectors σ, δ, α and $\sigma + \delta$ are pairwise linearly dependent. If we have simultaneously $\sigma \prec u\alpha$ and $\delta \prec v\alpha$, we obtain the contradictory inequality $m\alpha + \beta = \sigma + \delta \prec (u+v-1)\alpha + \beta = m\alpha + \beta$. Hence, by

Lemma 4, in (1.5) we need to consider only the following pairs of vectors σ and δ: 1) $\sigma = u\alpha$, $\delta = (v-1)\alpha + \beta$; 2) $\sigma = (u-1)\alpha + \beta$, $\delta = v\alpha$. In order to conclude the proof it remains to use the symmetry of (1.5) with respect to σ and δ. The lemma is proved.

Let us transform (1.9):

$$i(\omega, m\alpha+\beta)S_{m+1}^{m\alpha+\beta} = \sum_{\substack{u+v=m \\ u>0, v\geq 0}} \frac{\langle \alpha, v\alpha+\beta\rangle}{i(\omega, v\alpha+\beta)} u S_u^{u\alpha} i(w, v\alpha+\beta) S_{v+1}^{v\alpha+\beta}. \tag{1.10}$$

We introduce the following notation:

$$x_m = mS_m^{m\alpha}, \quad y_{m+1} = i(\omega, m\alpha+\beta)S_{m+1}^{m\alpha+\beta}, \quad l_v = \frac{\langle \alpha, v\alpha+\beta\rangle}{i(\omega, v\alpha+\beta)}.$$

Then (1.10) may be written as

$$y_{m+1} = \sum_{\substack{u+v=m \\ u>0, v\geq 0}} l_v x_u y_{v+1}.$$

LEMMA 6.

$$y_{m+1} = a_m x_1^m y_1, \tag{1.11}$$

where

$$a_0 = 1, \quad a_m = \sum_{\substack{u+v=m \\ u>0, v\geq 0}} u K_u h^{u-1} a_v l_v, \quad h = \frac{\langle \alpha, \alpha\rangle}{i(\omega, \alpha)}.$$

The proof is by induction over m, applying (1.7).

Let us set $uK_u = r_u$. From (1.8) we obtain

$$r_1 = 1, \qquad r_m = \sum_{\substack{u+v=m \\ u>0, v>0}} \frac{r_u r_v}{2}. \tag{1.12}$$

Taking the new notation into account, we have

$$a_m = \sum_{\substack{u+v=m \\ u>0, v\geq 0}} r_u h^{u-1} a_v l_v. \tag{1.13}$$

LEMMA 7.

$$1 - \sqrt{1-2z} = \sum_{n=1}^{\infty} r_n z^n.$$

COROLLARY. $r_m = (2m-3)!!/m!!$ *for* $m > 1$.

PROOF OF LEMMA 7. From (1.12) it follows that the power series $f(z) = \sum_1^\infty r_n z^n$ satisfies $f^2 - 2f + 2z = 0$. Since $f(0) = 0$, we have $f(z) = 1 - \sqrt{1-2z}$, as required.

From (1.13) we obtain successively

$$a_1 = r_1 l_0, \qquad a_2 = r_2 h l_0 + r_1^2 l_0 l_1,$$
$$a_3 = r_3 h^2 l_0 + r_1 r_2 h l_0 l_1 + r_1 r_2 h l_0 l_2 + r_1^3 l_0 l_1 l_2, \ldots.$$

LEMMA 8. *For* $m \geq 1$

$$a_m = \sum_{0=j_0<j_1<\cdots<j_k<m} r_{j_1-j_0} r_{j_2-j_1} \cdots r_{m-j_k} h^{m-k-1} l_{j_0} l_{j_1} \cdots l_{j_k}. \tag{1.14}$$

This is easily derived from (1.13) by induction over m.

Let us proceed to analyze the secular set. Since the vectors α and β by assumption are linearly independent, the hyperplanes $(\omega, \alpha) = 0$ and $\Gamma_m = \{x: (\omega, m\alpha + \beta) = 0\}$ do not coincide. According to Lemma 1 the functions S_r^σ are analytic almost everywhere on Γ_m for $r < m + 1$. In order to find out whether the hyperplane Γ_m belongs to the secular set $\mathbf{B}_{m+1}$ it is necessary to study the inequality $y_{m+1} \neq 0$. Let us use (1.11). In it we have $x_1 = S_1^\alpha$ and $y_1 = i(\omega, \beta)S_1^\beta$. The coefficients S_1^α and S_1^β are nonzero according to (1.4) and the definition of the vertices α and β. If $(\omega, \beta) \equiv 0$ on the hyperplane Γ_m, then the vectors α and β must be collinear. However, this is not so. Hence, $x_1 \neq 0$ and $y_1 \neq 0$. Therefore $y_{m+1} \neq 0$ if and only if $a_m \neq 0$. Let us consider two cases: $\langle \alpha, \alpha \rangle = 0$ and $\langle \alpha, \alpha \rangle \neq 0$. In the first case we have $h = 0$ and (by Lemma 8) $a_m = l_0 l_1 \cdots l_{m-1}$. Since, according to assumption (0.11), $\langle \alpha, \beta \rangle \neq 0$, then all $l_s \neq 0$ and, consequently, $a_m \neq 0$. In the second case we introduce the number $\lambda = \langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle$. At the points of the hyperplane Γ_m we have $(\omega, \beta) = -m(\omega, \alpha)$, and therefore

$$l_v = \frac{\lambda + v}{v - m} h.$$

Since in our case $h \neq 0$, it follows from (1.14) that $a_m = 0$ if and only if λ is a root of the polynomial

$$P_m(x) = \sum_{0 = j_0 < j_1 < \cdots < j_k < m} r_{j_1 - j_0} \cdots r_{m - j_k} \frac{(x + j_0) \cdots (x + j_k)}{(j_0 - m) \cdots (j_k - m)}. \tag{1.15}$$

LEMMA 9.

$$P_m = \frac{(-1)^m}{m!} x \left(x + \frac{1}{2}\right) \cdots \left(x + \frac{m-1}{2}\right). \tag{1.16}$$

In order to prove Lemma 9 we consider the new polynomials

$$Q_n(y) = \left. \frac{P_n(x)}{-x} \right|_{x = n - y}, \qquad Q_0 = -\frac{1}{y}. \tag{1.17}$$

LEMMA 10. *The following recurrence relation holds*:

$$mQ_m = \sum_{\substack{u + v = m \\ u > 0, v \geq 0}} r_u (v - y) Q_v. \tag{1.18}$$

PROOF. In (1.15) we make the change $m - j_l = i_{k-l+1}$. Then

$$P_m(x) = -\frac{x}{m} r_m + \sum_{0 < i_1 < \cdots < i_k < m} r_{m - i_k} r_{i_k - i_{k-1}} \cdots r_{i_1} \frac{x}{-m} \frac{x + m - i_k}{-i_k} \cdots \frac{x + m - i_1}{-i_1}.$$

Singling out the summation over i_k, we obtain

$$P_m(x) = \frac{x}{-m} \sum_{i_k = 0}^{m+1} P_{i_k}(x) r_{m - i_k}, \qquad P_0 \equiv 1.$$

This relation may be rewritten as follows:

$$m \frac{P_m}{x} = \sum_{k=0}^{m-1} (k - m - x) \frac{P_k}{x + m - k} r_{m-k}.$$

Setting $x + m = y$ and $P_n = (n - y)Q_n$, we obtain

$$mQ_m(y) = \sum_{k=0}^{m-1} r_{m-k}(k - y) Q_k(y), \qquad Q_0 = -\frac{1}{y},$$

which is equivalent to (1.18).

LEMMA 11.

$$\sum_{n=0}^{\infty} Q_n z^n = -\frac{1}{y}\left(\frac{1+\sqrt{1-2z}}{2}\right)^{2y}. \tag{1.19}$$

PROOF. We set $g(z) = \sum_0^\infty Q_n z^n$. Then (1.18) leads to the following differential equation for g:

$$z\frac{dg}{dz} = \left(z\frac{dg}{dz} - yg\right) f.$$

Here f is the function of Lemma 7. Solving this linear differential equation with the initial condition $g(0) = -1/y$, we obtain

$$g(z) = -\frac{1}{y}\left(\frac{1+\sqrt{1-2z}}{2}\right)^{2y},$$

as required.

The function (1.19) is analytic for small z. Let us find its Maclaurin series. We set $g(z) = F(\varphi^{-1}(z))$, where

$$F(z) = -\frac{1}{y}\left(\frac{1+z}{2}\right)^{2y}, \qquad \varphi = \frac{1-z^2}{2}.$$

Since $\varphi'(1) \neq 0$, we may use Bürmann's theorem [13]:

$$g(z) = g(0) + \sum_{m=1}^{\infty} \frac{z^m}{m!}\frac{d^{m-1}}{dz^{m-1}}\bigg|_{z=1} [F'(z)\Psi^m(z)], \tag{1.20}$$

where $\Psi = (z-1)/\varphi(z) = -2/(1+z)$. From (1.19) and (1.20) we easily obtain

$$m!Q_m = \left(\frac{2m-1}{2} - y\right)\left(\frac{2m-2}{2} - y\right)\cdots\left(\frac{m+1}{2} - y\right).$$

Returning to the old variable x and using (1.17), we obtain (1.16) for the polynomial $P_m(x)$. Lemma 9 is proved.

Let us continue the analysis of the secular set. Lemma 9 yields $a_m \equiv 0$ on the hyperplane Γ_m if and only if $\lambda = \langle\alpha, \beta\rangle/\langle\alpha, \alpha\rangle$ coincides with one of the following numbers: $0, -\frac{1}{2}, -1, \ldots, -(m-1)/2$. However, according to assumption (0.11), $\lambda \neq -m/2$ for all integers $m \geq 0$. Hence, the hyperplane $\Gamma_m = \{x: \langle x, m\alpha + \beta\rangle = 0\}$ belongs to the secular set $\mathbf{B}_{m+1} \subset \mathbf{B}$. For $m \to \infty$ the hyperplanes Γ_m accumulate, clearly, at the limit plane $\langle x, \alpha\rangle = 0$. The proof of the Main Lemma is finished.

§2. Proof of Theorems 2 and 3

Assume that the n analytic functions

$$F^{(k)} = \sum_{s=0}^{\infty} F_s^{(k)}(x, y)\varepsilon^s, \qquad k = 1, \ldots, n, \tag{2.1}$$

are first integrals of a Hamiltonian system with Hamiltonian (0.2). All functions $F_s^{(k)}$ are of course 2π-periodic with respect to the variables $y_1, \ldots, y_n$.

LEMMA 12. *The functions $F_0^{(1)}, \ldots, F_0^{(n)}$ do not depend on the angular variables $y_1, \ldots, y_n$, and at the points of the secular set their Jacobian*

$$\frac{\partial(F_0^{(1)}, \ldots, F_0^{(n)})}{\partial(x_1, \ldots, x_n)} \tag{2.2}$$

vanishes.

Theorem 2 follows immediately from the Main Lemma of §1 and Lemma 12. In fact, since the set $\mathbf{B} \subset \mathbf{R}^n$ consists of infinitely many different hyperplanes passing through the origin, then $\mathbf{B}$ is a uniqueness set for the class of functions analytic in $\mathbf{R}^n$: any analytic function vanishing at the points of $\mathbf{B}$ is zero everywhere on $\mathbf{R}^n$. Hence, by Lemma 12, the analytic function (2.2) is identically zero. This in turn means that the integrals (2.1) are dependent for $\varepsilon = 0$.

The proof of Theorem 3 uses another auxiliary construction that goes back to Poincaré ([1], §81).

LEMMA 13. *Assume that the functions $F_0^{(1)}, \ldots, F_0^{(n-1)}$ are independent at some point $x_0 \in \Gamma$ and Hamilton's equations have an additional formal integral $F = \sum F_s(x, y)\varepsilon^s$ independent of the functions $F^{(1)}, \ldots, F^{(n-1)}$. Then there exist a neighborhood V of the point x_0 in $\mathbf{R}^n = \{x\}$ and a formal integral*

$$\Phi = \sum_{s=0}^{\infty} \Phi_s(x, y)\varepsilon^s$$

with analytic coefficients in $V \times \mathbf{T}^n$ such that the functions $F_0^{(1)}, \ldots, F_0^{(n-1)}$ and Φ_0 are independent in $V \times \mathbf{T}^n$.

The functions $F_0^{(1)}, \ldots, F_0^{(n-1)}$ and Φ_0 actually depend on the variables x, by Lemma 12. We do not give here the proof of Lemma 13 since it essentially repeats Poincaré's reasoning in [1], §81. Theorem 3 follows now from Lemma 12, the Main Lemma, and the fact that the intersection $\mathbf{B} \cap V$ is a uniqueness set for the class of analytic functions in the domain V.

Lemma 12 generalizes the well-known assertion of Poincaré on the dependence of the functions $F_0^{(1)}, \ldots, F_0^{(n)}$ on the set $\mathbf{B}_1$ (see [1], §82). We discuss the proof of Lemma 12.

The first part of the lemma on the independence of the functions $F_0^{(k)}$ of the angular variables $y_1, \ldots, y_n$ was proved by Poincaré in [1], §82. The second part may be deduced, for instance, from a result of [4] (Chapter II, §4.3), which we state here as an auxiliary assertion.

LEMMA 14. *Assume that Hamilton's equations with Hamiltonian* (0.2) *have n first formal integrals $F^{(1)}, \ldots, F^{(n)}$ such that*

i) *the $F_0^{(k)}$ depend only on $x_1, \ldots, x_n$, and*

ii) *the Jacobian* (2.2) *is nonzero at all points of the domain $D \subset \mathbf{R}^n$.*

Then there exists a generating function $S = \sum_{m \geq 0} S_m(u, y)\varepsilon^m$ of the classical scheme of perturbation theory whose coefficients are analytic in the direct product $D \times \mathbf{T}^n$.

We deduce Lemma 12 from this. If the Jacobian (2.2) is nonzero at some point $x_0 \in \mathbf{B}$, it is nonzero in a whole neighborhood V of this point. According to Lemma 14 in the direct product $V \times \mathbf{T}^n$ one can (at least formally) construct the series of perturbation theory in powers of ε with analytic coefficients. However, by construction of the secular set $\mathbf{B}$ at the points of $\{x_0\} \times \mathbf{T}^n \subset V \times \mathbf{T}^n$ at least one of the functions S_m, $m = 1, 2, \ldots$, is not analytic.

§3. Proof of Theorem 1

Let us make the canonical transformation $x, y \mapsto x', y'$ by the formulas $x' = (B^T)^{-1}x$, $y' = By$, where B is an integer unimodular matrix. In the new variables the Hamiltonian $H_0 + H_1$ has the same form, and the set $\mathfrak{M}$ is transformed into

$\mathfrak{M}' = \{m'\}$, where $m' = (B^T)^{-1}m$. Since the integer vectors m are transformed in the same way as the impulses x, the fulfillment of the integrability condition (0.11) may be verified in the original variables. In fact, let a and b be vectors of $\mathbf{Z}^n$, and let a' and b' be their images under the map $m \mapsto (B^T)^{-1}m$. Then

$$\langle a', b' \rangle' = (BAB^T a', b') = (Aa, b) = \langle a, b \rangle.$$

Let us first prove Corollary 2 of Theorem 1.

LEMMA 15. *Let α be a vertex of the polyhedron $\mathscr{E}(\mathfrak{M})$, let Γ be an adjacent edge, and let β be the point of $\mathfrak{M} \cap \Gamma$ that is nearest to α. There exists an integer unimodular matrix B such that under the map $m \mapsto m' = (B^T)^{-1}m$ the points α and β are mapped into vertices of $\mathfrak{M}'$.*

The proof is based on an inductive application of the following well-known fact: for every integer vector $k_1 = (k_1^1, \ldots, k_1^m)$ with relatively prime coordinates there exist $m - 1$ additional integer vectors $k_2, \ldots, k_m$ such that $\det \|k_j^l\| = \pm 1$. Let l be the greatest common divisor of the components of the vector $\alpha - \beta$, and let B_1 be an integer unimodular matrix of dimension $n \times n$ whose lower row consists of the components of the vector $(\alpha - \beta)/l$. Under the map

$$m \to f(m), \qquad f(m) = (B_1^T)^{-1}m$$

the vector $(\alpha - \beta)/l$ is transformed into $e_n = (0, \ldots, 0, 1)^T$.

Now let us project the convex polyhedron $\mathscr{E}(\mathfrak{M}')$, $\mathfrak{M}' = f(\mathfrak{M})$, onto the hyperplane generated by the basis vectors $e_1, \ldots, e_{n-1}$. Then the edge $\Gamma' = f(\Gamma)$ is projected onto a vertex of the resulting convex polyhedron. Let us consider an edge Δ adjoining this vertex. Using a suitable integer unimodular matrix B_2 of dimension $(n-1) \times (n-1)$, one can make the edge Δ parallel to the $(n-1)$st coordinate axis. Let us repeat this operation $n - 2$ times. One can check that

$$B_1 \left\| \begin{array}{c|c} B_2 & 0 \\ \hline 0 & 1 \end{array} \right\| \cdots \left\| \begin{array}{c|ccc} B_n & & 0 & \\ \hline 0 & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{array} \right\|$$

is the desired matrix. The lemma is proved.

LEMMA 16. *Let α and β be neighboring vertices of the polyhedron $\mathscr{E}(\mathfrak{M})$. If the Hamiltonian system is completely integrable, the angle between the vectors α and β is not less than $\pi/2$.*

This follows directly from Lemma 15 and Theorem 2.

LEMMA 17. *Assume that a convex polyhedron in $(\mathbf{R}^n, \langle\ ,\ \rangle)$ is symmetric with respect to the origin and the angle between the radius-vectors of any two neighboring vertices is not less than $\pi/2$. Then this polyhedron is a rhomboid.*

We carry out the proof by induction on the dimension of the polyhedron M. When $\dim M = 1$ the assertion is clearly valid. Assume that the conclusion of the lemma is true for $\dim M \leq m$. Let α be one of the vertices of an $(m+1)$-dimensional polyhedron, and let Π_α be the closed half-space in $\mathbf{R}^{m+1}$ not containing α whose boundary $\partial\Pi_\alpha$ passes through the origin orthogonally to the vector α. By assumption all vertices of M that are joined with α by a vertex belong to Π_α. Actually all vertices of M except α lie in Π_α. Indeed, assume that there is a vertex β not belonging to Π_α. The convex polyhedron M is the union of the set M_α, the convex hull of all

vertices except α, and the set R_α, the convex hull of the one-dimensional edges of M adjoining α. The vertex β clearly does not lie in R_α. The segment Γ joining α and β lies entirely in the convex polyhedron M. However, Γ has only one point in common with R_α, the point α, since otherwise $\Gamma \subset R_\alpha$, and therefore the point β would not be a vertex of M. On the other hand, the segment Γ does not lie entirely in M_α, as otherwise $M = M_\alpha$. We have reached a contradiction. Analogously, all vertices of M except $(-\alpha)$ lie in $\Pi_{-\alpha}$. Thus, M is the convex hull of the points $\alpha, -\alpha$ and the convex hull of the remaining vertices of the polyhedron M lying in $\partial\Pi_\alpha$. The latter is a rhomboid by the inductive hypothesis. The lemma is proved.

Let us proceed to prove Theorem 1.

LEMMA 18. *Let P be a hyperplane in $\mathbf{R}^n$, and assume that the points of the set $P \cap \mathbf{Z}^n \subset \mathbf{R}^n$ form a subgroup of $\mathbf{Z}^n$ of rank $n-1$. Then there exists an integer unimodular matrix B whose last $n-1$ columns (or rows) are vectors of $P \cap \mathbf{Z}^n$.*

The proof is easily deduced from well-known results on the structure of the subgroups of $\mathbf{Z}^n$ (see, for instance, [14], Chapter VII).

Let α be one of the vertices of the rhomboid $\mathscr{E}(\mathfrak{M})$ and let Π_α be the closed subspace mentioned in the proof of Lemma 3. The intersection $\partial\Pi_\alpha \cap \mathbf{Z}^n$ is a subgroup of $\mathbf{Z}^n$ whose rank equals $\dim \mathscr{E}(\mathfrak{M}) - 1$. Let us complement this subgroup (if necessary) to a subgroup of rank $n-1$ so that the vector α does not belong to it. By Lemma 18 there exists a matrix B whose last $n-1$ rows are vectors of this subgroup, and the first row—a vector of $\mathbf{Z}^n$—has a positive projection onto α in the metric $\langle\ ,\ \rangle$. After the canonical change of coordinates $x \mapsto (B^T)^{-1}x$, $y \mapsto By$ we have that:

i) the first coordinate of each vector $\tau \in \partial\Pi_\alpha \cap \mathbf{Z}^n$ equals zero,

ii) the first coordinate of the vector α is positive,

iii) the vector α is the maximal element of $\mathfrak{M}$ (with respect to the standard order relation $\prec$ in $\mathbf{Z}^n$), and

iv) if the vector τ does not lie in Π_α, then $0 \prec \tau$.

LEMMA 19. *If system (0.1) is completely integrable, then all points of $\mathfrak{M}$ not lying in Π_α belong to the segment Γ joining the points 0 and α.*

Assume the contrary. By property iii) the vector α is a vertex of the set $\mathfrak{M}$. Let β be the vertex of $\mathfrak{M}$ adjoining α. By our assumption and the definition of adjoining vertex, β does not lie either in the half-space Π_α or on the segment Γ. Since α and β belong to the same half-space $\mathbf{R}^n \backslash \Pi_\alpha$, the inner product $\langle \alpha, \beta \rangle$ is positive. Hence, condition (0.11) holds and according to Theorem 2 the system of Hamilton's equations (0.1) is nonintegrable. This contradiction proves the lemma.

Applying Lemma 19 to all vertices of the rhomboid $\mathscr{E}(\mathfrak{M})$, we obtain Theorem 1.

§4. Generalizations

1. The condition of nondegeneracy of the quadratic form $H_0 = (Ax, x)/2$ in Theorem 2 may be replaced by the following weaker conditions:

i) $Am \neq 0$ for all integer vectors $m \neq 0$.

ii) The vectors $A\alpha$ and $A\beta$ are linearly independent.

We note that if $\det A = 0$ conditions i) and ii) can hold simultaneously only for $n \geq 3$.

We give a simple example. If

$$A = \begin{Vmatrix} 1 & \sqrt[3]{2} & \sqrt[3]{4} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad \alpha = (1,0,0)^T, \quad \beta = (1,-1,0)^T,$$

then the matrix A is degenerate, but conditions i) and ii) hold.

2. We note that Theorem 2 is not valid in the case when the Fourier coefficients of the perturbing function H_1 depend on x. We give an instructive counterexample:

$$H = a^2x_1^2 + abx_1x_2 + b^2x_2^2 + \frac{\varepsilon}{ax_1 - bx_2}(\sin y_1 - \sin y_2).$$

A Hamiltonian system with this Hamiltonian function is integrated by the method of separation of variables: the analytic functions

$$F_1 = a^3x_1^3 - ax_1H + \varepsilon \sin y_1, \qquad F_2 = b^3x_2^3 - bx_2H + \varepsilon \sin y_2$$

constitute a complete set of independent integrals.

In this problem we have $\alpha = (1,0)^T$ and $\beta = (0,1)^T$, and therefore inequality (0.11) becomes $b/a \neq -m/2$ for every integer $m \geq 0$. The "limit" line $\langle \alpha, x\rangle = 2ax_1 + bx_2 = 0$ does not coincide with the line $ax_1 - bx_2 = 0$ in whose points the Hamiltonian is not defined (cf. Theorem 3). However, integrability holds for all values of the ratio b/a (irrational values included).

Let $\alpha', \alpha'', \ldots$ be elements of the set $\mathfrak{M}$ lying between the vertices α and β (with respect to the lexicographic order $\prec$ in $\mathbf{Z}^n$). Clearly, each vector $\alpha', \alpha'', \ldots$ is linearly independent with α. Modifying the reasonings in §1 one can prove Theorems 2 and 3 also in the case when the coefficients $h_\alpha, h_{\alpha'}, h_{\alpha''}, \ldots, h_\beta$ are constant (here the remaining Fourier coefficients may be nonconstant analytic functions of $x_1, \ldots, x_n$).

3. If the perturbing function H_1 is not a trigonometric polynomial, the problem on the existence of additional integrals of a Hamiltonian system is simplified considerably: the nonintegrability of the perturbed system is established, as a rule, after finitely many steps of the perturbation theory.

To fix the ideas, let us consider the case of two degrees of freedom. Assume, then, that $H_0 + \varepsilon H_1$, where

$$H_0 = \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}x_ix_j, \quad H_1 = \sum_{m\in\mathbf{Z}^2} h_m e^{i(m,y)}, \qquad h_m = \text{const}.$$

If the secular set $\mathbf{B}_1$ consists of finitely many lines on the plane $\mathbf{R}^2 = \{x_1, x_2\}$, the nonintegrability of the perturbed Hamiltonian system follows from the generalized Poincaré theorem [3]. We therefore assume that the number of "resonance" lines making up the set $\mathbf{B}_1$ is finite. On the set $(\mathbf{R}^2\backslash\mathbf{B}_1) \times \mathbf{T}^2$ it is possible to carry out the first step of perturbation theory. The integrability of the perturbed system depends now on the structure of the secular set $\mathbf{B}_2$. Let us describe this set. To this end, let us consider the trigonometric series

$$\sum h_k'(x)e^{i(k,y)}, \qquad h_k' = \sum_{\tau+\sigma=k} \frac{\langle \tau, \sigma\rangle h_\tau h_\sigma}{\langle x, \tau\rangle\langle x, \sigma\rangle}. \tag{4.1}$$

The coefficients of this series are defined in $\mathbf{R}^2\backslash\mathbf{B}_1$. Using (1.2), it is easy to show that the set $\mathbf{B}_2$ consists of the points $x \in \mathbf{R}^2\backslash\mathbf{B}_1$ satisfying the following conditions:

i) $\langle x, k\rangle = 0$ for some $k \in \mathbf{Z}^2$, $k \neq 0$.

ii) $h_k'(x) \neq 0$.

In a typical situation the set $\mathbf{B}_2$ contains infinitely many different lines passing through the origin. This in turn implies the nonexistence of a formal integral with analytic coefficients in $\mathbf{R}^2 \times \mathbf{T}^2$ (see §2).

4. As an example of application of the remarks in this section, we prove the following fact.

PROPOSITION 2. *Assume that $n = 2$ and the secular set $\mathbf{B}_1$ consists of two lines altogether. Then Hamilton's equations have an additional formal integral if and only if these lines are orthogonal (in the metric $\langle\ ,\ \rangle$).*

PROOF. *Sufficiency.* Since $\mathbf{B}_1$ consists of two lines, then in (4.1) $\tau = \lambda\tau_0$ and $\sigma = \mu\sigma_0$, where $\tau_0, \sigma_0 \in \mathbf{Z}^2$, and λ and μ are integers. The orthogonality of the lines constituting the set $\mathbf{B}_1$ means the orthogonality of the vectors τ_0 and σ_0. Let us show that in this case the Hamiltonian system can be *integrated* by the method of separation of variables. In fact, let $\tau_0 = (\tau_1, \tau_2)$ and $\sigma_0 = (\sigma_1, \sigma_2)$. We set

$$Y_1 = (\tau_1 y_1 + \tau_2 y_2), \qquad Y_2 = (\sigma_1 y_1 + \sigma_2 y_2).$$

This homogeneous transformation of the angular coordinates is uniquely extended to a homogeneous canonical transformation $x, y \mapsto X, Y$. In the new variables we have

$$H = \tfrac{1}{2}(A_{11}X_1^2 + 2A_{12}X_1X_2 + A_{22}X_2^2) + \varepsilon(f(Y_1) + g(Y_2)),$$

where $A_{ij} = \text{const}$, and f and g are analytic 2π-periodic functions. Since $\langle\tau_0, \sigma_0\rangle = 0$, then $A_{12} = 0$. Consequently, the Hamiltonian system has two integrals that are linear in ε:

$$\tfrac{1}{2}A_{11}X_1^2 + \varepsilon f(Y_1), \qquad \tfrac{1}{2}A_{22}X_2^2 + \varepsilon g(Y_2).$$

Necessity. Let us consider first the case when the perturbing function H_1 is a trigonometric polynomial. As the edges of the set $\mathfrak{M}$ we may take the vectors $\alpha = \pm\lambda_*\tau_0$ and $\beta = \pm\mu_*\sigma_0$, where λ_* and μ_* are positive integers. Assume that $\langle\alpha, \alpha\rangle \geq 0$ (the case $\langle\alpha, \alpha\rangle \leq 0$ is considered analogously). If $\langle\alpha, \beta\rangle \neq 0$, then with no loss of generality we may assume that $\langle\alpha, \beta\rangle > 0$ (otherwise we replace β by $-\beta$). But then $m\langle\alpha, \alpha\rangle + 2\langle\alpha, \beta\rangle > 0$ for all integers $m \geq 0$. Therefore, if the Hamiltonian system has an additional integral, by the theorem we have $2\langle\alpha, \beta\rangle = 0$. This condition is clearly equivalent to $\langle\tau_0, \sigma_0\rangle = 0$.

Now let us consider the remaining case when H_1 is not a polynomial. We use the remarks in subsection 3. Since τ_0 and σ_0 are linearly independent, for a fixed $k = \lambda\tau_0 + \mu\sigma_0$ the numbers λ and μ are uniquely determined. By assumption, H_1 is not a polynomial and hence there are infinitely many different numbers λ and μ. If $\langle\tau_0, \sigma_0\rangle \neq 0$, from (4.1) it follows that $\mathbf{B}_2$ consists of infinitely many different lines, and therefore it is a uniqueness set for the class of analytic functions in $\mathbf{R}^2 = \{x_1, x_2\}$. To conclude the proof of the nonexistence of a formal integral it remains to use the reasoning in §2. The proposition is proved.

5. Proposition 2 may be generalized to systems with $n > 2$ degrees of freedom (with certain corrections). Let us assume that all points of the set $\mathfrak{M}$ lie on $l \leq n$ lines passing through the origin, and their directional vectors are linearly independent. Then one may assert that the Hamiltonian system with Hamiltonian function $H_0 + \varepsilon H_1$ has n single-valued analytic integrals that are independent for all sufficiently small values of ε if and only if these l lines are pairwise orthogonal (in the metric $\langle\ ,\ \rangle$). For $l = 1$ the system is clearly integrable.

As an example let us consider the Hamiltonian system with Hamiltonian

$$H = \frac{1}{2}\sum_{s=1}^{n} x_s^2 + \varepsilon[f(y_1 - y_2) + \cdots + f(y_{n-1} - y_n)], \tag{4.2}$$

where f is a real analytic 2π-periodic function. This system describes the dynamics of a "nonperiodic" chain of n particles on the line. It turns out that if $n > 2$ and $f \neq \text{const}$, then the system with Hamiltonian (4.2) has no complete set of independent integrals. In fact, in this case $l = n - 1$ and the corresponding lines are

determined by the vectors $(1,-1,0,\dots,0)^T,\dots,(0,\dots,1,-1)^T$, which are not all pairwise orthogonal. If we "close" the chain by adding to the Hamiltonian (4.2) the term $\varepsilon f(y_n - y_1)$, our assertion is no longer applicable: $l = n$ lines lie in the hyperplane orthogonal to the vector $(1,\dots,1)^T$. The integrability of the "periodic" chain depends essentially on the concrete form of the interaction potential f.

6. Let us indicate yet another way to generalize Theorem 2. We are concerned with the existence of analytic complex-valued integrals as functions of real canonical variables $(x, y) \in \mathbf{R}^n \times \mathbf{T}^n$. In the expressions (0.2) and (0.3) for the Hamiltonian function the coefficients a_{ij} and h_m are assumed complex numbers (here it is completely unnecessary to require that h_m and h_{-m} be complex conjugate). It is not difficult to see that Theorem 2 is valid in this more general situation.

We illustrate the idea by the example of generalized Toda chains (see [15]). The class of Hamiltonian systems we are interested in are systems with interaction of exponential type with Hamiltonian function

$$H = \frac{1}{2}\sum_{s=1}^{n} x_s^2 + \varepsilon \sum_{k=1}^{l} c_k e^{(a_k, y)}, \tag{4.3}$$

where $l \le n$, $c_k = \text{const} \neq 0$, and $a_1,\dots,a_l$ is a set of linearly independent vectors in $\mathbf{R}^n$. The systems with Hamiltonians (4.3) are not included in our considerations due to the nonperiodicity of the exponent. However, if we make the linear canonical transformation $y \mapsto iy$, $x \mapsto x/i$, we obtain a Hamiltonian system with Hamiltonian function

$$H = -\frac{1}{2}\sum x_s^2 + \varepsilon \sum c_k e^{(ia_k, y)}. \tag{4.4}$$

Making another linear transformation $y \mapsto Y$ by the formulas

$$Y_s = (a_s, y), \quad s \le l, \qquad Y_s = y_s, \quad s > l,$$

and extending it to a canonical transformation of all the phase space, we obtain a Hamiltonian function of the form (0.2) that is 2π-periodic with respect to $Y_1,\dots,Y_n$. To this system we can already apply Theorem 2.

It is known (see [15] and [16]) that if the vectors $a_1,\dots,a_l$ form a system of simple roots of a simple Lie algebra, then Hamilton's equations with Hamiltonian (4.3) are completely integrable. One can prove that the independent commuting integrals represented in the new variables are periodic with respect to the Y variables. Little is known on the integrability of the generalized Toda chain in the general case. We now find necessary conditions for the existence of a complete set of integrals periodic in Y. As we shall see, these conditions will actually lead us to systems of simple roots. Since $l \le n$, then (in the variables x, y) we may choose as the vertices α and β of the set $\mathfrak{M} = \{a_1,\dots,a_n\}$ any pair of vectors a_j and a_k $(j \neq k)$. By Theorem 2 a necessary integrability condition is the following (see (0.11)): the quantities $2(a_j, a_k)/(a_j, a_j) = 2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle$ are nonpositive integers. Thus, the matrix

$$\left\| \frac{2(a_j, a_k)}{(a_j, a_j)} \right\| \tag{4.5}$$

is the Cartan matrix of some root system. However, it is unclear whether this condition is sufficient for the integrability of the generalized Toda chain. [15] and [16] do not give a complete answer to this question.

O. I. Bogoyavlenskiĭ noted earlier a necessary condition for the integrability of a system with Hamiltonian (4.3) in an informal sense [15]. It consists in the finiteness

of the Coxeter group generated by the reflections with respect to the hyperplanes orthogonal to the vectors a_l.

In [9] a criterion was found for algebraic integrability of a system with Hamiltonian function (4.3), where $l = n + 1$. Under the assumption that each n vectors of the collection $a_1, \ldots, a_{n+1}$ are linearly independent, a condition for algebraic integrability is that (4.5) be a Cartan matrix.

7. As we observed in the Introduction, an obstruction to the complete integrability of the Hamiltonian system (0.1) is the collapse of the invariant tori $\mathbf{R}^n = \{y \bmod 2\pi,\ x = x^0\}$ of the unperturbed system lying on the hyperplanes $\langle m\alpha + \beta, x^0\rangle = 0$. Let us consider in greater detail the problem on the bifurcations of these resonance tori for systems with two degrees of freedom.

PROPOSITION 3. *Let α and β be vectors of $\mathfrak{M}$ satisfying the assumptions of Theorem 2, and let $x^0 \neq 0$ be a point in $\mathbf{R}^2$ lying on one of the lines $\langle m\alpha + \beta, x^0\rangle = 0$, $m = 0, 1, 2, \ldots$. Assume that the components of the integer vector $m\alpha + \beta$ are relatively prime. Then for small $\varepsilon \geq 0$ the Hamiltonian system with Hamiltonian function $H_0 + \varepsilon H_1$ has two periodic solutions analytic in ε such that*

(1) *their trajectories lie at a fixed positive level of the energy integral,*

(2) *for $\varepsilon = 0$ they coincide with a pair of periodic solutions lying on the invariant torus $\mathbf{T}_0^2 = \{y \bmod 2\pi,\ x = x^0\}$,*

(3) *their characteristic exponents $\pm\mu$ can be expanded in a convergent series in powers of $\sqrt{\varepsilon}$, and*

$$\mu = \mu_0(\sqrt{\varepsilon})^{m+1} + o((\sqrt{\varepsilon})^{m+1}), \qquad \mu_0 \neq 0,$$

(4) *one of these solutions is elliptic ($\mu_0^2 < 0$) and the other is hyperbolic ($\mu_0^2 > 0$).*

For small fixed values of $\varepsilon > 0$ Proposition 3 guarantees the existence in the general case of a large (but finite) number of different nondegenerate periodic solutions (for which $\mu \neq 0$). As $\varepsilon \to 0$, their number increases indefinitely. Using the inverse change of variables with respect to (0.5) this assertion may be restated for a system with Hamiltonian $H_0 + H_1$: for every large value $h > 0$ at the level of the energy integral $H_0 + H_1 = h$ there are many nondegenerate solutions with small period. As $h \to \infty$ their number increases without bound. It is known that on the trajectories of nondegenerate periodic solutions the first integrals are dependent (see [2] and [4]). From this fact and Proposition 3 one can deduce Theorem 2 (cf. [4]). The proof of Proposition 3 relies upon the application of a generalized version of Poincaré's theorem on the generation of periodic solutions that is proved in [17].

In the multidimensional case one must consider the problem of bifurcation of families of $(n-1)$-dimensional tori with incommensurable frequencies, into which the typical resonance tori of the unperturbed system can be fibered. It would be desirable to find a multidimensional analogue of Proposition 3.

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