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SOLUTION BRANCHING AND POLYNOMIAL INTEGRALS IN AN INVERTIBLE SYSTEM ON A TORUS

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We consider the motion of a point over the torus $\mathbf{T}^n = \{x \bmod 2\pi\}$ with a flat metric under the influence of a periodic force, whose components F_s are meromorphic functions over $\mathbf{C}^n$. Let $x = az + b, z \in \mathbf{C}$ be a line in $\mathbf{C}^n$, and let f_s be meromorphic functions of the variable z obtained by restricting F_s to this line. If the f_s have poles with nonzero residues, then the general solution of the equations of motion is nonsingular in the plane of complex time (Theorem 1). Suppose the meromorphic vector-function f has m poles with linearly independent residues, and the equations of motion allow k independent single-valued first integrals which are polynomials in velocities. Then $m + k \le n$ (Theorem 2). This result solves the problem of the relation between the single-valuedness of a general solution of the equations of motion and the integrability of the equations (Painlevé–Golubev problem).

1. Let $\mathbf{T}^n = \{x^1, \ldots, x^n \bmod 2\pi\}$ be the phase space of a mechanical system with n degrees of freedom

$$T = \frac{1}{2} g_{jk}\dot{x}^j\dot{x}^k, \quad g_{jk} = \text{const}$$

the kinetic energy, which is a covector field on $\mathbf{T}^n$ and

$$F = \{F_1, \ldots, F_n\}$$

the force acting on the system. The equations of motion

$$g_{kj}\ddot{x}^j = F_k \ (k = 1, \ldots, n) \tag{1}$$

are reversible: for each solution $t \mapsto x(t)$ there exists a solution $t \mapsto x(-t)$. All known integrals of system (1) are either polynomials in velocities $\dot{x}^1, \ldots, \dot{x}^n$ with unique coefficients over $\mathbf{T}^n$ or functions of polynomials.

Suppose that the components of the force F_s are analytical over $\mathbf{T}^n$ and extend to meromorphic functions over the affine space of complex variables $x^1, \ldots, x^m$. Then we can treat system (1) as a system of differential equations over $\mathbf{C}^n$ with complex time $t \in \mathbf{C}$. We are interested in relating the single-valuedness of the general solution of system (1) to the existence of $k \le n$ single-valued polynomial integrals.

An integral polynomial in velocities will be called a single-valued function if its coefficients are

1) periodic in $x^1, \ldots, x^n$ with a period of 2π and;

2) holomorphic over the region $\mathbf{C}^n \setminus \mathscr{P}$ where $\mathscr{P}$ is the union of polar sets of meromorphic functions $F_1, \ldots, F_n$.

Let

$$x = az + b, \quad a, b \in \mathbf{C}^n, \quad z \in \mathbf{C} \tag{2}$$

be a line in the complex space $\mathbf{C}^n$. Suppose that the restrictions of meromorphic functions $F_1, \ldots, F_n$ onto this line are meromorphic functions over the complex plane $z \in \mathbf{C}$. We denote

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them by $f_1, \ldots, f_n$. Functions f_s are automatically meromorphic if the line (2) transversally intersects the polar set $\mathcal{P}$ at points which are not ambiguity points of functions F_s. Suppose $\mathcal{P}$ is a complex hypersurface in $\mathbf{C}^n$ and the set of ambiguity points is of complex codimension two, the above property holds for almost all values of a and b. An aggregate of meromorphic functions $f_1, \ldots, f_n$ forms a vector-function f. Thus we can speak of residues of the function f at its poles. The residues are vectors in $\mathbf{C}^n$, they clearly depend on the choice of a and b.

THEOREM 1. Suppose that for some $a, b \in \mathbf{C}^n$ a function $z \mapsto f(z)$ has a pole with a nonzero residue. Then the general solution of system (1) is not a single-valued function of complex time.

THEOREM 2. Suppose that

1) for some $a, b \in \mathbf{C}^n$ a function f has m poles at which the residues are linearly independent over the field $\mathbf{C}$, and

2) system (1) has k single-valued polynomial integrals with almost-everywhere-independent leading uniform forms.

Then $m + k \leq n$.

Let us consider a simple example. Let n = 1 and $F(x) = \operatorname{sn}(2Kx/\pi, \varkappa)$ where K is a complex elliptic integral with modulus $\varkappa > 0$. Since f has simple poles, we can apply Theorems 1 and 2. Thus the general solution is multivalued and the equations of motion do not have a single-valued polynomial integral. Note that over the real domain there exists a single-valued polynomial integral, namely the energy integral. Over the complex phase space, however, this function has logarithmic singular points. The problem of nonexistence of polynomial integrals of Eqs. (1) for real values of x is considerably more complicated.

Suppose the forces have potentials ($F_s = -\partial V/\partial x^s$) and potential V is a periodic meromorphic function. Then Eqs. (1) permit an energy integral, which is a single-valued polynomial function. Thus $m \leq n - 1$. It is easy to give examples of force fields with a potential for which $m = n - 1$.

2. Theorem 1 follows from the following fact:

Proposition 1. Suppose that for some a, b the function $z \mapsto f_j(z)$ has a pole at the point z_0 with a residue ζ_j. Then for sufficiently large values of $|\alpha|$, $\alpha \in \mathbf{C}$, a solution of system (1) with initial conditions $x(0) = b$, $\dot{x}(0) = \alpha a$ extends to a certain annular neighborhood of the point $t_0 = z_0/\alpha$, and upon circling the point t_0 the velocity $\dot{x}_s$ changes by the amount $2\pi i \alpha^{-1} g^{sj} \zeta_j + o(\alpha^{-1})$.

To prove the proposition we rewrite system (1) in the form of a first-order system of equations

$$\dot{x}^s = v^s, \ \dot{v}^s = g^{sj} F_j \quad (1 \leqslant s \leqslant n) \tag{3}$$

and make the substitution $v^s = \alpha u^s$, $t = \tau/\alpha$. If we denote differentiation by τ by a prime sign, then from (3) we obtain the following equations:

$$x^{s\prime} = u^s, \ u^{s\prime} = \varepsilon g^{sj} F_j, \ \varepsilon = \alpha^{-2}. \tag{4}$$

Let ε be a small parameter, and let the line $x = a\tau + b$ be a solution of the unperturbed system. To complete the proof one simply needs to use Cauchy's theorem on residues and Poincaré's theorem on the decomposition of solutions to Eqs. (4) into convergent series in powers of ε.

To prove Theorem 2 we use the following:

LEMMA 1. The highest uniform form of a single-valued integral of Eqs. (1) is independent of variables x.

In fact, the highest uniform form of a polynomial integral is an integral of equations of motion on the n-dimensional torus $\mathbf{T}^n = \{x \bmod 2\pi\}$ with no forces present. Let x represent real angular coordinates. Then the real and imaginary parts of the uniform form are integrals of equations $\dot{x}^s = 0$. Since $\dot{x}^s = \text{const}$ and almost all trajectories of this system are everywhere dense on $\mathbf{T}^n$ any real periodic integral depends only on velocities $\dot{x}^s$. Q.E.D.

LEMMA 2. Let the conditions of Proposition 1 be satisfied, and let $\Phi(v^1, \ldots, v^n)$, $v^s = \dot{x}^s$, be the highest uniform form of a single-valued integral. Then

$$g^{sj}\frac{\partial\Phi}{\partial v^s}\zeta_j\equiv 0. \tag{5}$$

The proof uses the constancy of an integral over branching solutions which follows from Proposition 1. Substituting $v^s = \alpha u^s$ transforms the polynomial integral of system (3) into an integral of system (4), which is analytic in $1/\alpha$:

$$\Phi(u)+\frac{1}{\alpha}\Psi(u,x)+\ldots .$$

This function is invariant with respect to the substitution

$$u^s\mapsto(u^s)'=u^s+2\pi i\alpha^{-2}g^{sj}\zeta_j+o(\alpha^{-2}),$$
$$x^s\mapsto(x^s)'=x^s+O(\alpha^{-2}).$$

Therefore,

$$\Phi(u')+\frac{1}{\alpha}\Psi(u',x')+\ldots=\Phi(u)+\frac{1}{\alpha}\Psi(u,x)+\ldots .$$

Differentiating this relation with respect to α, multiplying both sides of α^3, and then letting $\alpha\to\infty$ gives relation (5).

Geometrically, condition (5) means that the gradient Φ'_V is orthogonal to the residue ζ. If the system has m independent residues and k integrals with independent gradients of highest forms, then, clearly, $m + k \leq n$. This completes the proof of Theorem 2.

3. It is well known that not every solution branching results in the absence of single-valued polynomial integrals. Theorems 1 and 2 give sufficient conditions for which the trajectories of branching solutions of Eqs. (1) are not confined to equipotential surfaces of polynomial integrals. This result solves the problem of the relation between the multivaluedness of a general solution to the system of Eqs. (1) and its integrability. The formulation of this problem is based on that of Painlevé. In [1] Golubev related this problem to the results of Kovalevskaya, Lyapunov, and Hoisson in the area of the dynamics of a heavy rigid body with a stationary point. Recently the Painlevé–Golubev problem was discussed in a number of articles related to physics [2, 3]. The role which branching solutions play in preventing integrability in the complex phase space was clarified for the first time in [4] using Poincaré's small-parameter method. Later Ziglin determined the relation between these problems and the self-crossing of complex separatrices [5], as well as the construction of the monodromy group of variational equations [6]. Another approach uses a well-known method of Lyapunov. The results of [5, 6] cannot be used for this problem. The proof of Theorems 1 and 2 is based on the methods of [4].

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