THE LIOUVILLE PROPERTY OF INVARIANT MEASURES OF COMPLETELY INTEGRABLE SYSTEMS AND THE MONGE–AMPÈRE EQUATION

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1. Liouville Invariant Measures. Let M^{2n} be a 2n-dimensional manifold, and let v be a smooth vector field which generates a dynamical system

$$\dot{x} = v(x), \quad x \in M. \tag{1}$$

Recall that this system is said to be Hamiltonian is there exist a closed nondegenerate 2-form Ω on M and a function H: $M \to R$ such that

$$\Omega(v, \cdot) = -dH. \tag{2}$$

The form Ω is usually called a symplectic structure, and the function H is called a Hamiltonian. Let us remark that a dynamical system can be represented in various ways in Hamiltonian form. By Liouville's theorem, the 2n-form

$$\Omega^n = \Omega \wedge \cdots \wedge \Omega$$

is invariant under the phase flow of system (1).

Definition. An invariant 2n-form μ of system (1) is said to be a Liouville form if $\mu = \Omega^n$, where Ω is the symplectic structure appearing in Eq. (2).

The problem of whether a given invariant measure is a Liouville measure includes the problem of recognizing the Hamiltonian nature of a dynamical system and hence, in the general case, is hopelessly complex. However, for some classes of dynamical systems it admits a constructive solution. One such class is that of completely integrable systems, given by equations

$$\dot{I}_1 = \cdots = \dot{I}_n = 0, \quad \dot{\varphi}_1 = \omega_1, \ldots, \dot{\varphi}_n = \omega_n. \tag{3}$$

Here $I = (I_1, \ldots, I_n)$ are coordinates in some domain $D \subset R^n$, $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a set of angle coordinates, which parametrize the points of the n-dimensional torus T^n, and $\omega_1, \ldots, \omega_n$ are functions of I. The motion in the system (3) takes place on the n-dimensional tori I = const and is quasi-periodic with frequencies $\omega_1, \ldots, \omega_n$. In what follows we shall consider the nondegenerate case, in which

$$\frac{\partial(\omega_1, \ldots, \omega_n)}{\partial(I_1, \ldots, I_n)} \neq 0. \tag{4}$$

Our main result is that any invariant measure of equations (3) with a continuous positive density is Liouville. The proof of this result reduces to the solvability of a higher-dimensional analogue of the Monge-Ampère equation.

Unless otherwise stipulated, all objects encountered below are assumed to be infinitely differentiable.

2. Hamiltonian Property of Completely Integrable Systems. First let us consider the problem of representing the system (3) in the form of Hamilton's equations. For simplicity we shall assume that the domain D is a small neighborhood of some point at which inequality (4) holds.

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THEOREM 1. The system (3) is always Hamiltonian, and

$$\Omega = d\sigma, \qquad \sigma = \sum a_k(I)dI_k + \sum \frac{\partial K}{\partial \omega_k} d\varphi_k, \tag{5}$$

$$H = \sum \omega_k \frac{\partial K}{\partial \omega_k} - K + \text{const}, \tag{6}$$

where $a_1, \ldots, a_n$ are arbitrary smooth functions on D, K is a function of $\omega_1, \ldots, \omega_n$, and

$$\det \left\| \frac{\partial^2 K}{\partial I_i \partial \omega_j} \right\| \neq 0. \tag{7}$$

Condition (7) guarantees that the 2-form Ω is nondegenerate. As a matter of fact, the functions a_k are not involved in the representation of equations (3) as Hamiltonian equations (because $\dot{I} = 0$). Hence, there are as many distinct Hamiltonian representations of Eqs. (3) as there are functions $K(\omega)$ satisfying condition (7).

The meaning of the function K is transparent: it is the Lagrangian of the system under consideration. Indeed, K is a function of the velocities $\dot{\varphi}_k = \omega_k$. The derivatives $\partial K/\partial \omega_k = I_k$ $(1 \leq k \leq n)$ are the momenta canonically conjugate to the coordinates φ_k. In the variables I, φ the symplectic structure (5) has the canonical form (the existence of which is given by Darboux's theorem)

$$d\Big(\sum \frac{\partial K}{\partial \omega_k} d\varphi_k\Big) = d \sum I d\varphi_k = \sum dI \wedge d\varphi_k.$$

Finally, formula (6) defines the Hamiltonian H in accordance with Legendre's transformation. The necessary condition for the existence of Legendre's transformation,

$$\det \left\| \frac{\partial^2 K}{\partial \omega^2} \right\| \neq 0,$$

is automatically satisfied thanks to (4) and (7).

Proof. For the sake of brevity, the proof of Theorem 1 will be carried out for the case n = 2. Set

$$\Omega = \alpha dI_1 \wedge dI_2 + \beta dI_1 \wedge d\varphi_1 + \gamma dI_1 \wedge d\varphi_2 + \delta dI_2 \wedge d\varphi_1 + \varepsilon dI_2 \wedge d\varphi_2 + \zeta d\varphi_1 \wedge d\varphi_2.$$

The coefficients $\alpha, \beta, \ldots, \zeta$ are smooth functions in $D \times T^2$. The condition that the form Ω be closed is equivalent to the four equations

$$\frac{\partial \alpha}{\partial \varphi_1} - \frac{\partial \beta}{\partial I_2} + \frac{\partial \delta}{\partial I_1} = 0, \quad \frac{\partial \alpha}{\partial \varphi_2} - \frac{\partial \gamma}{\partial I_2} + \frac{\partial \varepsilon}{\partial I_1} = 0,$$
$$\frac{\partial \beta}{\partial \varphi_2} - \frac{\partial \gamma}{\partial \varphi_1} + \frac{\partial \zeta}{\partial I_1} = 0, \quad \frac{\partial \delta}{\partial \varphi_2} - \frac{\partial \varepsilon}{\partial \varphi_1} + \frac{\partial \zeta}{\partial I_2} = 0. \tag{8}$$

Equality (2) yields four more equations:

$$\beta\omega_1 + \gamma\omega_2 = \frac{\partial H}{\partial I_1}, \quad \delta\omega_1 + \varepsilon\omega_2 = \frac{\partial H}{\partial I_2},$$
$$\zeta\omega_2 = \frac{\partial H}{\partial \varphi_1}, \quad \zeta\omega_1 = -\frac{\partial H}{\partial \varphi_2}. \tag{9}$$

The function H must be 2π-periodic in φ_1 and φ_2.

From the last two of Eqs. (9) we obtain the equality

$$\omega_1 \frac{\partial H}{\partial \varphi_1} + \omega_2 \frac{\partial H}{\partial \varphi_2} = 0. \tag{10}$$

Let us represent the Hamiltonian as a Fourier series

$$H = \sum H_{mn}(I) e_{mn}, \quad e_{mn} = \exp[i(m\varphi_1 + n\varphi_2)].$$

Then from (10) we obtain an infinite number of relations

$$(m\omega_1 + n\omega_2)H_{mn} = 0.$$

By the nondegeneracy assumption (4), the equality $m\omega_1 + n\omega_2 = 0$ cannot be satisfied in a whole neighborhood of any point of D. Consequently, $H_{mn} = 0$ for all m, n such that $m^2 + n^2 \neq 0$. This means that the Hamiltonian does not depend on the angles φ_1 and φ_2.

Since $\omega_1{}^2 + \omega_2{}^2 \neq 0$ on an everywhere dense subset of D, (9) implies that $\zeta = 0$.

Now let us apply the Fourier method to the last two of Eqs. (8):

$$n\beta_{mn} = m\gamma_{mn}, \quad n\delta_{mn} = m\varepsilon_{mn}. \tag{11}$$

Here β_{mn}, ... are the Fourier coefficients of the functions β, ... From (11) it follows that for $m^2 + n^2 \neq 0$,

$$\beta_{mn} = mA_{mn}, \quad \gamma_{mn} = nA_{mn};$$
$$\delta_{mn} = mB_{mn}, \quad \varepsilon_{mn} = nB_{mn}.$$

Applying Fourier's method to the first two of Eqs. (9) we obtain the relations

$$(m\omega_1 + n\omega_2)A_{mn} = 0, \quad (m\omega_1 + n\omega_2)B_{mn} = 0.$$

By the nondegeneracy assumptions, $A_{mn} = B_{mn} = 0$ for all $m^2 + n^2 \neq 0$. Therefore, the coefficients β, γ, ε, δ do not depend on the angles φ_1 and φ_2. From the first two of Eqs. (8) it follows that the same is true for the coefficient α.

Equations (8) now take the form

$$\frac{\partial \beta}{\partial I_2} = \frac{\partial \delta}{\partial I_1}, \quad \frac{\partial \gamma}{\partial I_2} = \frac{\partial \varepsilon}{\partial I_1}.$$

Therefore,

$$\beta = \frac{\partial F}{\partial I_1}, \quad \delta = \frac{\partial F}{\partial I_2}, \quad \gamma = \frac{\partial \Phi}{\partial I_2}, \quad \varepsilon = \frac{\partial \Phi}{\partial I_1}. \tag{12}$$

where F and Φ are some functions defined in D. Substituting the relations obtained in (9) we obtain the following two equation:

$$\frac{\partial F}{\partial I_1}\omega_1 + \frac{\partial \Phi}{\partial I_1}\omega_2 = \frac{\partial H}{\partial I_1}, \quad \frac{\partial F}{\partial I_2}\omega_1 + \frac{\partial \Phi}{\partial I_2}\omega_2 = \frac{\partial H}{\partial I_2}. \tag{13}$$

Equating the mixed derivatives of the function H, we arrive at an equation that connects the functions F and Φ:

$$\frac{\partial F}{\partial I_1}\frac{\partial \omega_1}{\partial I_2} + \frac{\partial \Phi}{\partial I_1}\frac{\partial \omega_2}{\partial I_2} = \frac{\partial F}{\partial I_2}\frac{\partial \omega_1}{\partial I_1} + \frac{\partial \Phi}{\partial I_2}\frac{\partial \omega_2}{\partial I_1}. \tag{14}$$

By the nondegeneracy condition (4), in a small neighborhood of D one can take ω_1 and ω_2 as independent variables. Accordingly, we shall consider that F and Φ are functions of ω. Then (14) takes on the following form:

$$\left(\frac{\partial F}{\partial \omega_2} - \frac{\partial \Phi}{\partial \omega_1}\right)\frac{\partial(\omega_1, \omega_2)}{\partial(I_1, I_2)} = 0.$$

Therefore, one can find a function $K(\omega_1, \omega_2)$ such that

$$F = \frac{\partial K}{\partial \omega_1}, \quad \Phi = \frac{\partial K}{\partial \omega_2}.$$

This conclusion in conjunction with formulas (12) yields formula (5).

Now let us find the Hamiltonian. Taking into account (13), we have

$$\frac{\partial H}{\partial I_k} = \omega_1 \frac{\partial}{\partial I_k}\frac{\partial K}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial I_k}\frac{\partial K}{\partial \omega_2} = \frac{\partial}{\partial I_k}\left(\omega_1 \frac{\partial K}{\partial \omega_1} + \omega_2 \frac{\partial K}{\partial \omega_2} - K\right),$$

$k = 1, 2$. This yields formula (6).

The theorem is proved.

3. The Monge-Ampère equation. Let

$$f(I, \varphi) dI_1 \wedge d\varphi_1 \wedge dI_2 \wedge d\varphi_2 \tag{15}$$

be an invariant measure of Eqs. (3) with a differentiable density f. This function satisfies the Liouville equation

$$\operatorname{div}(fv) = \omega_1 \frac{\partial f}{\partial \varphi_1} + \omega_2 \frac{\partial f}{\partial \varphi_2} = 0.$$

By the nondegeneracy condition, f does not depend on φ_1 and φ_2. As a matter of fact, this conclusion holds also for a measure (15) with a continuous density f.

On the other hand, by formula (5)

$$\Omega \wedge \Omega = \det \left\| \frac{\partial^2 K}{\partial I_i \partial \omega_j} \right\| dI_1 \wedge d\varphi_1 \wedge dI_2 \wedge d\varphi_2.$$

Hence, the problem of whether the invariant measure (15) is a Liouville measure reduces to that of the solvability with respect to K of the equation

$$\det \left\| \frac{\partial^2 K}{\partial I_i \partial \omega_j} \right\| = f(I). \tag{16}$$

As we already remarked in Sec. 2, as local coordinates I_1, I_2 we can take ω_1, ω_2. Then Eq. (16) takes on the form of the Monge–Ampère equation

$$\frac{\partial^2 K}{\partial I_1^2} \frac{\partial^2 K}{\partial I_2^2} - \left(\frac{\partial^2 K}{\partial I_1 \partial I_2} \right)^2 = f(I). \tag{17}$$

As is known, this equation is always solvable for any continuous positive function f. Moreover, it has many distinct solutions: they are parametrized by functions on the circle. Indeed, by [1], the Dirichlet problem for Eq. (17) is solvable if the boundary conditions for the function K are given on a convex curve in the domain D. On the other hand, according to an old result of Rellich, if $f > 0$, then the Dirichlet problem for the Monge-Ampère equation has no more than two distinct solutions.

In the case $n > 2$ the generalized Monge-Ampère equation (16) also has solutions for arbitrary continuous positive functions f. Rigorous formulations and the corresponding references concerning the Dirichlet problem for Eq. (16) can be found in the book [2].

Thus, we have proved

THEOREM 2. All invariant measures of a nondegenerate completely integrable system are Liouville measures.

4. Changes of Coordinates in Integrable Systems. Let us subject Eqs. (3) to the change of time variable $t \to \tau$

$$d\tau = F(I, \varphi) dt; \tag{18}$$

here F is a smooth positive function in $D \times T^n$. Then Eqs. (3) take on the form

$$I'_k = 0, \quad \varphi'_k = \omega_k F; \quad 1 \leq k \leq n. \tag{19}$$

Here the prime denotes differentiation with respect to τ.

A natural question is what are the conditions for the system (19) to be a Hamiltonian system. This problem was considered in the paper [3]. In classical dynamics there are known examples of nontrivial changes of time after which a completely integrable Hamiltonian system remains integrable, but with respect to a new symplectic form. In [3] the so-called Liouville changes of time (18) were introduced, with the property that one can find new angle coordinates $\psi_1, \ldots, \psi_n$, depending smoothly on I and φ, in which Eqs. (19) take the form

$$\psi'_k = \chi_k(I). \tag{20}$$

It is clear that the Liouville changes of time preserve the Hamiltonian character of a system, and that not all changes of time (18) are Liouville (see [3]).

THEOREM 3. If the system (19) is Hamiltonian and has no equilibrium positions, then the change of time (18) is Liouville.

Thus, the Liouville changes of time exhaust all changes of time under which the system remains Hamiltonian. For brevity we shall consider the case $n = 2$.

The conditions for the 2-form Ω to be closed are again of the form (8), while Eqs. (9) go into the equations

$$\begin{aligned} \beta\omega_1 + \gamma\omega_2 &= G\frac{\partial H}{\partial I_1}, & \delta\omega_1 + \varepsilon\omega_2 &= G\frac{\partial H}{\partial I_2}, \\ \zeta\omega_2 &= G\frac{\partial H}{\partial \varphi_1}, & \zeta\omega_1 &= -G\frac{\partial H}{\partial \varphi_2}, \end{aligned} \tag{21}$$

where $G = F^{-1}$. From the last two of Eqs. (21) it follows that

$$\omega_1\frac{\partial H}{\partial \varphi_1} + \omega_2\frac{\partial H}{\partial \varphi_2} = 0.$$

Hence, by the nondegeneracy assumption, the Hamiltonian depends only on I_1 and I_2. In particular, $\zeta = 0$ (see (21)), and relations (11) hold for the Fourier coefficients of the functions β, γ, δ, ε. Hence, we can again set

$$\beta_{mn} = mA_{mn}, \ldots, \varepsilon_{mn} = nB_{mn}.$$

From the first two equations of system (21) one obtains the equalities

$$\begin{aligned} (m\omega_1 + n\omega_2)A_{mn} &= \frac{\partial H}{\partial I_1}G_{mn}, \\ (m\omega_1 + n\omega_2)B_{mn} &= \frac{\partial H}{\partial I_2}G_{mn}, \end{aligned} \tag{22}$$

which are valid for $m^2 + n^2 \neq 0$. Here G_{mn} are the Fourier coefficients of the function G.

Since, by hypothesis, the system (3) has no equilibrium positions, $dH \neq 0$ in the domain D. Otherwise, the vector field v would vanish somewhere (because of the nondegeneracy of Ω). Consequently, from (22) one derives the equalities

$$G_{mn} = \Lambda_{mn}(m\omega_1 + n\omega_2), \qquad m^2 + n^2 \neq 0.$$

Let us introduce the function

$$R = -i\sum \Lambda_{mn}e_{mn}.$$

It obviously is 2π-periodic in φ_1, φ_2, and infinitely differentiable. It satisfies the equation

$$\frac{\partial R}{\partial \varphi_1}\omega_1 + \frac{\partial R}{\partial \varphi_2}\omega_2 = G - \langle G\rangle, \tag{23}$$

where

$$\langle C\rangle = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} G\, d\varphi_1\, d\varphi_2.$$

However, as it was established in [4, Chap. VII], if Eq. (23) admits a smooth single-valued solution, then one can indicate explicit formulas for a change to new angle coordinates ψ_1, ψ_2, under which Eqs. (19) for φ_1, φ_2 take the form (20). This change of coordinates depends smoothly on the coordinates I_1, I_2 as parameters. Hence, the change of coordinates (18) is Liouville.

In conclusion, let us note that the problem of periodic solutions of Eq. (23) is closely connected with the classical small divisors problem. The conditions for the reducibility of Eqs. (19) to the form (20) for n = 2 were first studied by A. N. Kolmogorov in [5]. They are connected with the existence of periodic solutions of the "homological equation," the discrete analogue of Eq. (23).

REFERENCES

1. A. V. Pogorelov, The Extrinsic Geometry of Convex Surfaces [in Russian], Nauka, Moscow (1969).
2. N. V. Krylov, Nonlinear Elliptic and Parabolic Equations of Second Order [in Russian], Nauka, Moscow (1985).
3. A. P. Veselov, "On changing the time in integrable systems," Vestn. Mosk. Gos. Univ., Ser. 1, Mat. Mekh., No. 5, 25-29 (1987).
4. V. V. Kozlov, Qualitative Analysis Methods in Rigid Body Dynamics [in Russian], Izd. Mosk. Gos. Univ. (1980).
5. A. N. Kolmogorov, "On dynamical systems with an integral invariant on the torus," Dokl. Akad. Nauk SSSR, **93**, No. 5, 763-766 (1953).