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POLYNOMIAL INTEGRALS OF GEODESIC FLOWS ON A TWO-DIMENSIONAL TORUS

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ABSTRACT. The geodesic curves of a Riemannian metric on a surface are described by a Hamiltonian system with two degrees of freedom whose Hamiltonian is quadratic in the momenta. Because of the homogeneity, every integral of the geodesic problem is a function of integrals that are polynomial in the momenta. The geodesic flow on a surface of genus greater than one does not admit an additional nonconstant integral at all, but on the other hand there are numerous examples of metrics on a torus whose geodesic flows are completely integrable: there are polynomial integrals of degree ≤ 2 that are independent of the Hamiltonian. It appears that the degree of an additional "irreducible" polynomial integral of a geodesic flow on a torus cannot exceed two. In the present paper this conjecture is proved for metrics which can arbitrarily closely approximate any metric on a two-dimensional torus.

Bibliography: 12 titles.

§1. INTRODUCTION. MAIN RESULT

Let Σ be a closed two-dimensional surface with Riemannian metric ds. In local isothermal coordinates q_1, q_2 on Σ this metric has the form

$$ds^2 = \frac{M}{2}(dq_1^2 + dq_2^2), \tag{1.1}$$

where M is a positive function of q_1 and q_2.

The geodesic curves of the Riemannian metric (1.1) are described by the Hamiltonian system

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}, \qquad k = 1, 2, \tag{1.2}$$

with Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2M}. \tag{1.3}$$

The phase space of this system is the total space of the cotangent bundle $T^*\Sigma$.

Let $F: T^*\Sigma \to \mathbb{R}$ be a smooth integral of the system (1.2): the Poisson bracket $\{H, F\}$ is identically equal to zero. We represent F as a formal Maclaurin series in p_1, p_2:

$$\sum_s F_s(p, q), \tag{1.4}$$

where F_s is a homogeneous form of degree s in the momenta p_1, p_2. We have a simple result.

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Proposition 1. *Every homogeneous form of the expansion* (1.4) *is an integral of the Hamiltonian equations* (1.2).

In particular, $s \geq 1$ (since there are no nonconstant integrals depending only on q_1 and q_2). Of course, not all of the polynomials F_s are independent. Proposition 1 reduces the problem of the integrability of a system with Hamiltonian (1.3) to a search for homogeneous polynomial integrals.

Remark. An example is given in [1] of a smooth metric on a closed surface Σ of arbitrary genus, whose geodesic flow admits a smooth integral F with zero Maclaurin series (1.4). This example must be related to the class of "pathological" examples: although the functions H and F are not everywhere dependent, nevertheless $F = 0$ in a whole domain of the phase space $T^*\Sigma$.

It is proved in [2] that if the genus of Σ is greater than 1, then the equations (1.2) do not admit a polynomial integral that is independent of H. Other proofs of this result are given in [3] and [4]. On the contrary, there are numerous examples of metrics on the sphere and torus whose geodesic flows admit additional integrals.

Definition. A polynomial integral of smallest degree, independent of the integral H, is said to be *irreducible.*

If a geodesic flow does not admit an additional polynomial integral, then in this case the degree of an irreducible integral can be assumed to be zero. It is clear that any integral of the Hamilton equations (1.2) is a function of an irreducible integral and the Hamiltonian H

Apparently the degree of an irreducible integral of the geodesic flow on an oriented surface of genus p does not exceed

$$4 - 2p. \tag{1.5}$$

For $p \geq 2$ this estimate is a result of [2]. For the torus ($p = 1$) we will find that the degree of an irreducible integral does not exceed 2. Examples are given in [5] of metrics on the sphere that admit integrals of degree 3 and 4.

The case $p = 1$ is considered below. It is well known that for a torus one can introduce global isothermal coordinates. Thus, we shall assume that q_1 and q_2 in (1.3) are the angular coordinates on a two-dimensional torus $\Sigma = \mathbb{T}^2$.

The estimate (1.5) for $p = 1$ is established in [5] under one of the following additional assumptions:

1) F is an even function of p_1 and p_2;

2) F is even in p_1 (resp., p_2) and odd in p_2 (resp., p_1).

We shall consider the case when the positive function M is a trigonometric polynomial. Let the finite sum

$$M = \sum [M]_{k_1 k_2} e^{i(k_1 q_1 + k_2 q_2)} \tag{1.6}$$

be its Fourier series. The spectrum of M is a finite subset of the integer lattice

$$S = \{k = (k_1, k_2) \in \mathbb{Z}^2 : [M]_{k_1 k_2} \neq 0\}.$$

It is mapped into itself under the map $k \to -k$.

Our main result is as follows:

Theorem 1. *Assume that Hamilton's equations* (1.2) *with Hamiltonian* (1.3) *admit an integral F which is polynomial in the momenta and independent of the integral H. Then*

1) *if the degree of F is even, then the spectrum S lies on two lines that intersect orthogonally at the origin;*

2) *if the degree of* F *is odd, then* S *lies on one line passing through the origin.*

In the first case we can superimpose the lines containing the spectrum S on the coordinate axes by a rotation around the origin. Then the Hamiltonian (1.3) will have the form

$$H = \frac{p_1^2 + p_2^2}{2[f(q_1) + g(q_2)]}. \tag{1.7}$$

Here f and g are periodic functions of one variable, whose sum is positive. The function (1.7) is the Hamiltonian of a Liouville system with a torus as configuration space. The variables q_1 and q_2 are separated: the Hamilton equations (1.2) with Hamiltonian (1.7) admit two quadratic integrals

$$\Phi_1 = \frac{p_1^2}{2} - Hf, \qquad \Phi_2 = \frac{p_2^2}{2} - Hg.$$

Since $\Phi_1 + \Phi_2 = 0$, these integrals are dependent. However, each of them is independent of the energy integral H.

In the second case we may assume that the line underlying the spectrum S coincides with the first coordinate axis. Then the angular coordinate q_2 will be cyclic: the Hamiltonian H does not depend on q_2. To this coordinate corresponds a linear cycle integral $\Phi = p_2$.

Birkhoff long ago noted (see [6], Chapter II) that to an integral, linear in the momenta, of a dynamical system with two degrees of freedom corresponds a hidden local cyclic coordinate, and the presence of an additional quadratic integral is related to the possibility of introducing local separated variables. Global versions of these results of Birkhoff are discussed in [3] and [7].

Taking this into consideration, from Theorem 1 we obtain:

Corollary. *If the equations of geodesics on a torus with metric* (1.1), *where* M *is a trigonometric polynomial, admit an additional polynomial integral* F, *then there is a polynomial integral of degree* ≤ 2 *that is independent of the function* H. *Moreover, if the degree of* F *is odd, then there exists a linear integral.*

Closely connected with the problem of homogeneous polynomial integrals of geodesic flows is the problem of polynomial integrals of the Hamiltonian system on $T^*\Sigma = \mathbb{R}^2 \times \mathbb{T}^2$ with Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2). \tag{1.8}$$

Here V is a smooth function on $\mathbb{T}^2$, the potential energy of a reversible mechanical system. The kinetic energy T is of course equal to $(p_1^2 + p_2^2)/2$. The integrals of this system that are polynomial in the momenta are not homogeneous:

$$F = F_n + F_{n-1} + F_{n-2} + \cdots + F_0. \tag{1.9}$$

Proposition 2. *Assume that a system with Hamiltonian* (1.8) *has a polynomial integral of degree* n, *independent of the function* (1.8). *Then a system with Hamiltonian* (1.3), *where* $M = h - V$, $h > \max V$, *admits a homogeneous polynomial integral of degree* $\leq n$.

We sketch the proof of Proposition 2 (cf. [5]). First of all we note that the Poisson bracket of two homogeneous polynomials in p_1 and p_2 of degree r and s, respectively, will be a homogeneous polynomial of degree $r + s - 1$. This gives us the following fact: if the sum (1.9) is an integral of a Hamiltonian system with Hamiltonian (1.8), then the functions

$$\Phi_1 = F_n + F_{n-2} + \cdots, \qquad \Phi_2 = F_{n-1} + F_{n-3} + \cdots$$

will also be integrals of this system. Finally, the polynomials

$$(1.10)\qquad \begin{gathered} F_n + F_{n-2}\left(\frac{T}{h-V}\right) + F_{n-4}\left(\frac{T}{h-V}\right)^2 + \cdots, \\ F_{n-1} + F_{n-3}\left(\frac{T}{h-V}\right) + \cdots, \end{gathered}$$

which are homogeneous in the momenta, are integrals of the geodesic flow on a torus with the metric

$$(1.11)\qquad \frac{p_1^2 + p_2^2}{2(h-V)}.$$

If the functions (1.8) and (1.9) are independent, then one of the polynomials (1.10) is independent of the Hamiltonian (1.11).

According to Proposition 2, if a system with the Hamiltonian (1.11) does not admit new polynomial integrals, then a system with Hamiltonian (1.8) will also not have an additional integral in the form of a polynomial in the momenta with unique coefficients on $\mathbb{T}^2$. Hence, the problem of the integrability of geodesic flows includes as a special case the problem of the integrability of reversible systems on a torus with a Hamiltonian of the form (1.8). The latter problem was studied in [8] and [9]. In [8] conditions for the existence of integrals of degree 3 and 4 were considered. It was shown that if there is a new integral of degree three, then there must be a linear integral. And if there is an integral of degree four, then there exists an independent integral of degree ≤ 2. In [9] Kozlov and Treshchëv considered multidimensional Hamiltonian systems on the m-dimensional Hamiltonian systems on the m-dimensional torus $\mathbb{T}^m = \{q_1, \dots, q_m \bmod 2\pi\}$ with a Hamiltonian of the form

$$\frac{1}{2}\sum_{i,j=1}^{m} a_{ij}p_ip_j + V(q_1, \dots, q_m),$$

where $\|a_{ij}\|$ is a positive definite matrix with constant coefficients and V is a function on $\mathbb{T}^m$. Existence conditions were studied for m independent integrals in the form of polynomials in the momenta p with unique coefficients on $\mathbb{T}^m$. It is not assumed a priori that the integrals are involutive. This problem has been solved completely for the case when the potential V is a trigonometric polynomial. It turned out that a complete set of polynomial integrals exists if and only if the spectrum of the polynomial V lies on $k \leq m$ lines which intersect pairwise orthogonally at the origin. In particular, there are m independent polynomial integrals of degree ≤ 2. We note that the problem of polynomial integrals of fixed degree is much simpler than the problem considered in [9].

Remark. A Riemannian metric is said to be nondegenerately integrable (see [10]) if its geodesic flow is integrable via a smooth integral, all of whose critical manifolds on a constant-energy surface are nondegenerate. Fomenko conjectured that any nondegenerately integrable metric on a two-dimensional torus is Liouville. This conjecture has not yet been proved. However, Nguyen, Polyakova, and Kalashnikov (Jr.) proved that this conjecture is true at least from the point of view of complexity of integrable metrics (the concept of complexity was introduced in [10]). In fact, the complexity of the geodesic flow of an arbitrary nondegenerately integrable smooth Riemannian metric on a two-dimensional torus is equal to the complexity of the geodesic flow of some Liouville metric.

§2. Integrals of odd degree

First we shall give some auxiliary results of a general nature. Let F_m be a homogeneous polynomial of p_1 and p_2 of degree m. We denote by F_m^* the value of this polynomial for $p_1 = 1$ and $p_2 = i$ (where $i^2 = -1$). If F depends on the angular coordinates q_1, q_2, then F^* is a function on $\mathbb{T}^2$.

Lemma 1 [3]. *The polynomial F is evenly divisible by the polynomial $H = (p_1^2 + p_2^2)/(2M)$ if and only if $F^* = 0$.*

Now let $m \geq 3$.

Lemma 2. *There is a polynomial F_{m-2} such that*

$$F_m = a_0 p_1 p_2^{m-1} + b_0 p_2^m + H \cdot F_{m-2}.$$

Indeed, we set

$$F_m^* = (i)^{m-1}(a_0 + b_0 i).$$

Then by Lemma 1 the homogeneous polynomial of degree m

$$F_m - a_0 p_1 p_2^{m-1} - b_0 p_2^m$$

is evenly divisible by H, as required.

Lemma 3. *A polynomial of degree $2n+1$ in the momenta can be represented in the following form*:

$$\begin{aligned} F_{2n+1} &= a_0 p_1 p_2^{2n} + b_0 p_2^{2n+1} \\ &\quad + H(a_1 p_1 p_2^{2n-2} + b_1 p_2^{2n-1}) + \cdots + H^n(a_n p_1 + b_n p_2), \end{aligned} \tag{2.1}$$

where $a_0, b_0, \ldots, a_n, b_n$ are smooth real functions on $\mathbb{T}^2$.

The proof is based on an inductive application of Lemma 2.

From now on we shall assume that F_{2n+1} is an integral of a geodesic flow on a torus.

Lemma 4. *F_{2n+1}^* is a holomorphic function of $z = q_1 + iq_2$.*

This assertion is a key moment in the well-known method of Birkhoff (see [6], Chapter II), associated with the introduction on Σ of a complex structure induced by the isothermal coordinates of the given Riemannian metric.

Birkhoff himself established that the function F^* is a holomorphic for integrals of degrees one and two. The proof for an integral of arbitrary degree can be found, for example, in [3].

Lemma 5. $a_0 + ib_0 = c_1 + ic_2 = \text{const}$.

Indeed, since the coefficients of F are periodic, the holomorphic function F^* is bounded. Hence, it is constant.

Now we shall set $a_0 = c_1$, $b_0 = c_2$. Since (2.1) is an integral of the Hamilton differential equations

$$\dot{q}_k = \Lambda p_k, \quad \dot{p}_k = \Lambda H \frac{\partial M}{\partial q_k}, \qquad k = 1, 2, \ \Lambda = \frac{1}{M},$$

it follows that

$$\dot{F}_{2n+1} = \Lambda H^{n+1}\left(a_n\frac{\partial M}{\partial q_1}+b_n\frac{\partial M}{\partial q_2}\right)$$
$$+\Lambda H^n\left(\frac{\partial a_n}{\partial q_1}p_1^2+\frac{\partial a_n}{\partial q_2}p_1p_2+\frac{\partial b_n}{\partial q_1}p_1p_2+\frac{\partial b_n}{\partial q_2}p_2^2\right) \tag{2.2}$$
$$+\cdots+\Lambda H\left(c_1\frac{\partial M}{\partial q_1}p_2^{2n}+2nc_1\frac{\partial M}{\partial q_2}p_1p_2^{2n-1}+(2n+1)c_2\frac{\partial M}{\partial q_2}p_2^{2n}\right)$$
$$\equiv 0.$$

From this relation we obtain a chain of partial differential equations satisfied by the coefficients of the polynomial (2.1):

$$\frac{\partial a_1}{\partial q_2}+\frac{\partial b_1}{\partial q_1}+2nc_1\frac{\partial M}{\partial q_2}=0,$$
$$-\frac{\partial a_1}{\partial q_1}+\frac{\partial b_1}{\partial q_2}+c_1\frac{\partial M}{\partial q_1}+(2n+1)c_2\frac{\partial M}{\partial q_2}=0, \tag{2.3.1}$$

$$\frac{\partial a_2}{\partial q_2}+\frac{\partial b_2}{\partial q_1}+2(n-1)a_1\frac{\partial M}{\partial q_2}=0,$$
$$-\frac{\partial a_2}{\partial q_1}+\frac{\partial b_2}{\partial b_2}+2M\frac{\partial a_1}{\partial q_1}+a_1\frac{\partial M}{\partial q_1}+(2n-1)b_1\frac{\partial M}{\partial q_2}=0, \tag{2.3.2}$$

....

$$\frac{\partial a_n}{\partial q_2}+\frac{\partial b_n}{\partial q_1}+2a_{n-1}\frac{\partial M}{\partial q_2}=0,$$
$$-\frac{\partial a_n}{\partial q_1}+\frac{\partial b_n}{\partial q_2}+2M\frac{\partial a_{n-1}}{\partial q1}+a_{n-1}\frac{\partial M}{\partial q_1}+3b_{n-1}\frac{\partial M}{\partial q_2}=0, \tag{2.3.n}$$

$$2M\frac{\partial a_n}{\partial q_1}+a_n\frac{\partial M}{\partial q_1}+b_n\frac{\partial M}{\partial q_2}=0. \tag{2.3.0}$$

The equations (2.3.1) are obtained from (2.2) after a preliminary cancellation by ΛH and the substitutions $p_1 = 1$, $p_2 = i$. We derive equations (2.3.2). For this we rewrite (2.2) after cancelling by ΛH and some regrouping of terms:

$$\{\cdots\}H+\frac{\partial a_1}{\partial q_1}p_1^2p_2^{2n-2}+\left(\frac{\partial a_1}{\partial q_2}+\frac{\partial b_1}{\partial q_1}+2nc_1\frac{\partial M}{\partial q_2}\right)p_1p_2^{2n-1}$$
$$+\left(\frac{\partial b_1}{\partial q_2}+c_1\frac{\partial M}{\partial q_1}+(2n+1)c_2\frac{\partial M}{\partial q_2}\right)p_2^{2n}\equiv 0. \tag{2.4}$$

We use the trivial identity

$$p_1^2p_2^{2n-2}=2MHp_2^{2n-2}-p_2^{2n}.$$

If we take the previously obtained equations (2.3.1) into account, relation (2.4) takes the following form:

$$\{\cdots\}H+2MH\frac{\partial a_1}{\partial q_1}p_2^{2n-2}\equiv 0.$$

Again cancelling by H and substituting $p_1 = 1$ and $p_2 = i$, we will obtain equations (2.3.2). The remaining equations of the chain (2.3) are derived in the same way.

We use Fourier's method to solve the system of equations (2.3). We set

$$a_m+\sum[a_m]_{u,v}e^{i(uq_1+vq_2)},\qquad b_n=\sum[b_m]_{u,v}e^{i(uq_1+vq_2)}.$$

Equations (2.3.1) are linear and thus are easily solved:

$$(2.5.1)\qquad (u^2+v^2)\begin{pmatrix}[a_1]_{u,v}\\ [b_1]_{u,v}\end{pmatrix}=\begin{pmatrix}u^2-2nv^2 & (2n+1)uv\\ -(2n+1)uv & -(2n+1)v^2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}[M]_{u,v}.$$

The equations (2.3.2) are already nonlinear. Therefore it is impossible to obtain simple formulas like (2.5.1) in this case. We shall use a method proposed in [9].

Let $\mathscr{E}(S)$ be the convex hull of the set S. It is a convex polygon, and the origin is its center of symmetry. Let (u, v) be one of the vertices of $\mathscr{E}(S)$. It is easy to see that the Fourier coefficients $[a_2]_{2u,2v}$ and $[b_2]_{2u,2v}$ are expressed just in terms of the uth and vth Fourier coefficients of the functions a_1, b_1, and M:

$$(2.5.2)\qquad \begin{aligned}&2(u^2+v^2)\begin{pmatrix}[a_2]_{2u,2v}\\ [b_2]_{2u,2v}\end{pmatrix}\\ &=\begin{pmatrix}3u^2-2(n-1)v^2 & (2n-1)uv\\ -(2n+1)uv & -(2n-1)v^2\end{pmatrix}\begin{pmatrix}[a_1]_{u,v}\\ [b_1]_{u,v}\end{pmatrix}[M]_{u,v}.\end{aligned}$$

This method allows us to obtain analogous formulas for the coefficients $[a_k]_{ku,kv}$ and $[b_k]_{ku,kv}$. For example, here are the explicit formulas for $k=n$:

$$(2.5.n)\qquad \begin{aligned}&n(u^2+v^2)\begin{pmatrix}[a_n]_{nu,nv}\\ [b_n]_{nu,nv}\end{pmatrix}\\ &=\begin{pmatrix}(2n-1)u^2-2v^2 & 3uv\\ -(2n+1)uv & -3v^2\end{pmatrix}\begin{pmatrix}[a_{n-1}]_{(n-1)u,(n-1)v}\\ [b_{n-1}]_{(n-1)u,(n-1)v}\end{pmatrix}[M]_{u,v}.\end{aligned}$$

Equation (2.3.0) gives one more relation:

$$(2.5.0)\qquad (2n+1)u[a_n]_{nu,nv}+b[b_n]_{nu,nv}=0.$$

Taking (2.5.1)–(2.5.n) and (2.5.0) into account, we obtain as a final result

(2.6)

$$\begin{aligned}&((2n+1)u,\ v)\begin{pmatrix}(2n-1)u^2-2v^2 & 3uv\\ -(2n+1)uv & -3v^2\end{pmatrix}\begin{pmatrix}(2n-3)u^2-4v^2 & 5uv\\ -(2n+1)uv & -5v^2\end{pmatrix}\\ &\cdots\begin{pmatrix}3u^2-(2n-2)v^2 & (2n-1)uv\\ -(2n+1)uv & -(2n-1)v^2\end{pmatrix}\begin{pmatrix}u^2-2nv^2 & (2n+1)uv\\ -(2n+1)uv & -(2n+1)v^2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=0.\end{aligned}$$

Lemma 6. *The left-hand side of* (2.6) *is equal to*

$$(2.7)\qquad (2n+1)!!(c_1\cos m\alpha+c_2\sin m\alpha),$$

where $m=2n+1$ *and* $\tan\alpha=v/u$.

This curious matrix identity is proved in the Appendix. It is clear that α is the angle between the ray passing through the vertex $(u, v)\in S$, and the horizontal coordinate axis.

Putting (2.6) and (2.7) together, we obtain

$$(2.8)\qquad \tan m\alpha=-\frac{c_1}{c_2}.$$

This relation obviously holds for all vertices of the convex hull of the spectrum S. The solutions of equation (2.8) are the m angles

$$\alpha,\ \alpha+\frac{\pi}{m},\ \alpha+\frac{2\pi}{m},\ \dots,\ \alpha+\frac{(m-1)\pi}{m},$$

measured from the horizontal axis. Hence, the polygon $\mathscr{E}(S)$ can have at most $2m$ vertices, located on the lines $l_1, \dots, l_m$, which pass through the origin and form the angles

$$(2.9)\qquad \frac{\pi}{m},\ \frac{2\pi}{m},\ \dots,\ \frac{(m-1)\pi}{m}$$

between themselves.

Lemma 7 (see, e.g., [11]). *Let* $0 < \beta < \pi/2$, $\beta \neq \pi/4$, *and* $\beta = p\pi/q$, *where* p *and* q *are natural numbers. Then* $\tan\beta$ *is an irrational number.*

Corollary. *If* m *is odd, then the tangents of the angles* (2.9) *are irrational.*

We now assume that the spectrum S does not lie on a single line. Then there are two distinct lines l_1 and l_2, passing through the origin, that contain two vertices of the convex hull $\mathscr{E}(S)$. The tangent of the angle between l_1 and l_2 is rational. In fact, the tangents of the angles which the lines l_1 and l_2 form with the coordinate axis are rational. It remains to use the well-known formula for the tangent of a difference. On the other hand, the angle between l_1 and l_2 is equal to one of the angles (2.9). According to the corollary of Lemma 7 the tangent of this angle is irrational if m is odd. This contradiction proves Theorem 1 for the case of an integral of odd degree.

§3. Integrals of even degree

Now suppose $m = 2n$. Using Lemmas 2 and 5, an integral of even degree can be represented in the form

$$\begin{aligned} F_n = c_1 p_1 p_2^{2n-1} + c_2 p_2^{2n} + H(a_1 p_1 p_2^{2n-3} + b_1 p_2^{2n-2}) \\ + \cdots + H^{n-1}(a_{n-1} p_1 p_2 + b_{n-1} p_2^2) + H^n b_n. \end{aligned} \tag{3.1}$$

Here c_1, $c_2 = \text{const}$, and $a_1, b_1, \dots, b_n$ are smooth functions on $\mathbb{T}^2$.

Setting the derivative $\dot{F}_{2n}$ equal to zero, we obtain a chain of equations for the coefficients of the polynomial (3.1):

$$\begin{gathered} \frac{\partial a_1}{\partial q_2} + \frac{\partial b_1}{\partial q_1} + (2n-1)c_1\frac{\partial M}{\partial q_2} = 0, \\ -\frac{\partial a_1}{\partial q_1} + \frac{\partial b_1}{\partial q_2} + c_1\frac{\partial M}{\partial q_1} + 2nc_2\frac{\partial M}{\partial q_2} = 0, \end{gathered} \tag{3.2.1}$$

$$\begin{gathered} \frac{\partial a_2}{\partial q_2} + \frac{\partial b_2}{\partial q_1} + 2(n-3)a_1\frac{\partial M}{\partial q_2} = 0, \\ -\frac{\partial a_2}{\partial q_1} + \frac{\partial q_2}{\partial q_2} + 2M\frac{\partial a_1}{\partial q_1} + a_1\frac{\partial M}{\partial q_1} + (2n-2)b_1\frac{\partial M}{\partial q_2} = 0, \end{gathered} \tag{3.2.2}$$

....

$$\begin{gathered} \frac{\partial b_n}{\partial q_1} + a_{n-1}\frac{\partial M}{\partial q_2} = 0, \\ \frac{\partial b_n}{\partial q_2} + 2M\frac{\partial a_{n-1}}{\partial q_1} + a_{n-1}\frac{\partial M}{\partial q_1} + 2b_{n-1}\frac{\partial M}{\partial q_2} = 0. \end{gathered} \tag{3.2.n}$$

Let (u, v) be a vertex of the convex hull of the spectrum S. Applying the method of §2, we obtain an equality analogous to (2.6):

$$((2n-1)u^2 - v^2, 2uv)\begin{pmatrix} (2n-3)u^2 - 3v^2 & 4uv \\ -2nuv & -4v^2 \end{pmatrix}\begin{pmatrix} (2n-5)u^2 - 5v^2 & 6uv \\ -2nuv & -6v^2 \end{pmatrix}$$
$$\cdots\begin{pmatrix} u^2 - (2n-1)v^2 & 2nuv \\ -2nuv & -2nv^2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0.$$

It can be transformed to the form (see the Appendix)

$$(2n-1)!!(c_1\cos m\alpha + c_2\sin m\alpha) = 0, \tag{3.3}$$

where $\tan\alpha = u/v$. Hence

$$\tan m\alpha = -\frac{c_1}{c_2} = \text{const}.$$

Consequently, in this case as well the vertices of the convex hull of S lie on m lines, which pass through the origin and form the angles (2.9) between themselves. For even $m \geq 4$ we have both $\pi/2$ and $\pi/4$ among them. Using Lemma 7, it is easy to deduce from this that for even m the polygon $\mathscr{E}(S)$ can have 2, 4, 6, or 8 vertices, and its principal diagonals (passing through the origin) are either orthogonal or intersect at an angle $\pi/4$. This result is of independent interest, but, of course, we cannot derive the conclusion of Theorem 1 from it.

We use the concept of adjoining vertex, introduced in [9]. For this we consider the standard lexicographic ordering in $\mathbb{R}^2$: we shall say that (k_1, k_2) is greater than (s_1, s_2) if one of the following two conditions holds:

$$1)\ k_1 > s_1, \qquad 2)\ k_1 = s_1,\ k_2 > s_2.$$

Let $\alpha = (u, v)$ be a maximal element of the spectrum S. This point is clearly one of the vertices of the convex polygon $\mathscr{E}(S)$. A maximal vector $\beta \in S$ that is linearly independent of α is called an *adjoining vertex* to α. If S does not lie on a single line, then an adjoining vertex trivially exists. Otherwise, the Hamilton equations admit a linear integral. Our task is to prove that the vectors α and β are orthogonal.

Lemma 8 [12]. *Let $s\alpha + \beta = \tau_1 + \dots + \tau_{s+1}$, where $\tau_i \in S$. Then $\tau_k = \beta$ and $\tau_j = \alpha$ for all $j \neq k$.*

First we consider the special case when the largest element $\alpha = (u, v)$ of S lies on the horizontal coordinate axis, so that $u > 0$, $v = 0$. Let $\beta = (k, l)$ be an adjoining vertex. We show that $k = 0$. Since S is invariant under a reflection relative to the origin, it will follow from this that the points of S lie on the coordinates lines.

Since $v = 0$, the angle α in (3.3) is equal to zero. Hence the constant $c_1 = 0$. Clearly $c^2 \neq 0$. Otherwise, by Lemmas 1 and 5, the integral is evenly divisible by H.

Solving the system (3.2.1) by Fourier's method, we obtain

$$[a_1]_\alpha = [b_1]_\alpha = 0,$$

(3.4.1)
$$[a_1]_\beta = \frac{mklc_2}{k^2 + l^2}[M]_\beta, \qquad [b_1]_\beta = -\frac{ml^2c_2}{k^2 + l^2}[M]_\beta.$$

From (3.2.2) we derive the equalities

$$[a_2]_{2\alpha} = [b_2]_{2\alpha} = 0.$$

Using them, and also applying Lemma 8, we obtain

(3.4.2)
$$\begin{pmatrix} [a_2]_{\alpha+\beta} \\ [b_2]_{\alpha+\beta} \end{pmatrix} = \begin{pmatrix} (2k+u)(k+u) \\ -l(2k+u) \end{pmatrix} \frac{[a_1]_\beta [M]_\alpha}{l^2 + (k+u)^2}.$$

Similarly one deduces formulas for the Fourier coefficients $[a_s]_{\beta+(s-1)\alpha}$, $[b_s]_{\beta+(s-1)\alpha}$. We write them down explicitly for $s = n - 1$:

$$[a_{n-1}]_{(n-1)\alpha} = [b_{n-1}]_{(n-1)\alpha} = 0,$$

(3.4.$n-1$)
$$\begin{pmatrix} [a_{n-1}]_{\beta+(n-2)\alpha} \\ [a_{n-1}]_{\beta+(n-2)\alpha} \end{pmatrix} = \begin{pmatrix} (k+(n-2)u)(2k+(2n-5)u) \\ -l(2k+(2n-5)u) \end{pmatrix} \frac{[a_{n-2}]_{\beta+(n-3)\alpha}[M]_\alpha}{l^2 + (k+(n-2)u)^2}.$$

Finally, from the last system (3.2.n) we derive the equalities

(3.4.n)
$$[b_n]_{\beta+(n-1)\alpha} = 0,$$
$$(2k + (2n-3)u)[M]_\alpha [a_{n-1}]_{\beta+(n-2)\alpha} = 0.$$

Since $m = 2n \geq 4$, we have $n \geq 2$. Since β is an adjoining vertex, $k > 0$. Hence $2k + (2n-3)u > 0$, and it follows form the last equality of (3.4.n) that

$$[a_{n-1}]_{\beta+(n-2)\alpha} = 0. \tag{3.5}$$

For $n = 2$ the coefficients $a_2, b_2, \ldots$ are not defined. It follows from (3.5) that $[a_1]_\beta = 0$. Since $l \neq 0$, $c_2 \neq 0$, and $[M]_\beta \neq 0$, we obtain the desired equality $k = 0$ from (3.4.1).

If $n \geq 3$, then $wk + (2n-5)u > 0$, and from equalities (3.4.2)–(3.4.$n-1$) we successively find

$$[a_{n-2}]_{\beta+(n-3)\alpha} = 0, \ldots, [a_1]_\beta = 0.$$

But then from (3.4.1) it again follows that $k = 0$, as required.

We turn to the general case. Let $\alpha = (u, v)$ be a vertex of $\mathscr{E}(S)$ at greatest distance from the origin. If there are several such vertices, we take one of them. We set

$$A = -\begin{pmatrix} u/s & v/s \\ u/s & u/s \end{pmatrix}, \qquad s = \sqrt{u^2+v^2}.$$

This is an orthogonal matrix; it defines a rotation of the plane through the angle $\alpha = \arctan v/u$. We consider a linear transformation of the plane $\mathbb{R}^2 = \{q_1, q_2\}$:

$$q' = A^T q.$$

It can be extended to a canonical transformation $(q, p) \to (q', p')$ if we set

$$p' = A^{-1} p.$$

Since A is orthogonal, the Hamiltonian (1.3) has the same form in the new variables p', q'. In the denominator we will have the function $M(q)$, where $q = (A^T)^{-1} q' = Aq'$. Clearly $M'(q') = M(Aq')$ is periodic with period $2\pi/s$ in the new coordinates. Hence, it can be expanded in a Fourier series. This series will have exactly as many harmonics as the Fourier series (1.6) of the function $M(q_1, q_2)$. The spectrum S' of the trigonometric polynomial M' is obviously obtained from the spectrum S via the linear transformation defined by the matrix A. Since the vector $A\alpha$ has components $(s, 0)$, the largest vertex of S' lies on the horizontal axis.

We note that the equations (3.2) are linear and homogeneous relative to the derivatives. Therefore the formulas (3.3) and (3.4), which are a consequence of (3.2), do not depend on the period of the function M. Thus, we are in the conditions of the special case considered above. Hence, the spectrum S' lies on the coordinate axes. Making the inverse transformation with the orthogonal matrix A^{-1}, we find that the points of the original spectrum S lie on two lines that intersect orthogonally at the origin. Thus Theorem 1 is completely proved.

Appendix. Matrix binomial identities

We set

$$B_m = \begin{bmatrix} (2m+1)x^2 - 2(n-m)y^2 & (2n-2m+1)xy \\ -(2n+1)xy & -(2n-2m+1)y^2 \end{bmatrix},$$

$$C_m = \begin{bmatrix} (2m+1)x^2 - (2n-2m-1)y^2 & 2(n-m)xy \\ -2nxy & -2(n-m)y^2 \end{bmatrix}.$$

Theorem 2.

(1) $$[(2n+1)x, y]B_{n-1}\cdots B_1 B_0 \begin{bmatrix} 1 \\ i \end{bmatrix} = (2n+1)!!(x+iy)^{2n+1},$$

(2) $$[(2n-1)x^2 - y^2, 2xy]C_{n-2}\cdots C_0 \begin{bmatrix} 1 \\ i \end{bmatrix} = (2n-1)!!(x+iy)^{2n}.$$

The conclusion of Lemma 6 follows form the matrix binomial identity (1). For this it suffices to set $c_1 = 1$ and $c_2 = \pm i$ (where $i^2 = -1$). In turn, equality (3.3) is a consequence of identity (2). It is not inconceivable that the identities (1)–(2) have a nontrivial combinatorial interpretation.

Below we give a proof of Theorem 2, found by D. V. Treshchëv. For definiteness we consider identity (1). The left- and right-hand sides of (1) are homogeneous polynomials of degree $2n+1$ in x and y. Hence, it suffices to prove equality (1) for the case $x^2 + y^2 = 1$. With this goal in mind we set $x = \cos\alpha$ and $y = \sin\alpha$. Then (1) takes the form

(3) $$u_n A_{n-1}\cdots A_1 A_0 \begin{bmatrix} 1 \\ i \end{bmatrix} = (2n+1)!!e^{(2n+1)i\alpha},$$

where

$$u_n = \left[\frac{2n+1}{2}(e^{i\alpha} + e^{-i\alpha}), -\frac{i}{2}(e^{i\alpha} - e^{-i\alpha})\right],$$

$$A_m = \frac{1}{2}\begin{bmatrix} \frac{2n+1}{2}(e^{2i\alpha} + e^{-2i\alpha}) - (2n-1-4m) & -\frac{2n-2m+1}{2}i(e^{2i\alpha} - e^{-2i\alpha}) \\ \frac{2n+1}{2}i(e^{2i\alpha} - e^{-2i\alpha}) & \frac{2n-2m+1}{2}(e^{2i\alpha} + e^{-2i\alpha} - 2) \end{bmatrix}$$

We set

$$v_1 = 2A_0\begin{bmatrix} 1 \\ i \end{bmatrix}, \qquad v_{k+1} = 2A_k v_k, \quad k = 1, \ldots, n-1.$$

In this notation equality (3) has the form

(4) $$2^{-n}u_n v_n = (2n+1)!!e^{(2n+1)i\alpha}.$$

We introduce further notation:

$$\begin{aligned}
&a_0^m = a_1^m = 1, && a_s^m = 2n\cdots(2n-s+2),\\
&p_0^m = 1, && p_s^m = (2n+1)a_s^m,\\
&b_0^m = b_1^m = 1, && b_s^m = (2n-2m+s)\cdots(2n-2m+2),\\
&q_0^m = 1, && q_s^m = b_s^m(2n-2m+1),\\
&f_0^m = f_m^m = 1, && f_s^m = (2n-2m+2s+1).
\end{aligned}$$

Here $s = 1, 2, \ldots, m-1$.

Lemma 9.

$$v_m = \begin{bmatrix} \sum_{k=0}^{m}(-1)^k \binom{m}{k} p_{m-k}^m q_k^m e^{2(m-k)i\alpha} \\ (2n+1)i\sum_{k=0}^{m}(-1)^k \binom{m}{k} a_{m-k}^m f_k^m b_k^m e^{2(m-k)i\alpha} \end{bmatrix}.$$

This assertion is not hard to prove by induction on m. It is interesting to note that v_m does not contain negative powers of the exponentials $e^{2i\alpha}$.

We now derive formula (4) from Lemma 9. We have

$$u_n v_n = \frac{2n+1}{2}\left\{\sum_{k=0}^{n}(-1)^k\binom{n}{k}p_{n-k}^n q_k^n[e^{(2(n-k)+1)i\alpha}+e^{(2(n-k)-1)i\alpha}]\right.$$
$$\left.+\sum_{k=0}^{n}(-1)^k\binom{n}{k}a_{n-k}^n f_k^n b_k^n[e^{(2(n-k)+1)i\alpha}+e^{(2(n-k)-1)i\alpha}]\right\}.$$

We denote the coefficient of $(-1)^k(2n+1)\exp((2(n-k)+1)i\alpha)$ in this expression by the symbol μ_k. Then (4) is equivalent to the series of equalities

$$\mu_0 = 2^n(2n-1)!!, \qquad \mu_1 = \mu_2 = \cdots = \mu_{n+1} = 0.$$

We write down an explicit expression for μ_k :

$$\mu_k = \frac{1}{2}\left(\binom{n}{k}p_{n-k}^n q_k^n - \binom{n}{k-1}p_{n+1-k}^n q_{k-1}^n\right.$$
$$\left.+\binom{n}{k}a_{n-k}^n f_k^n b_k^n + \binom{n}{k-1}a_{n+1-k}^n f_{k-1}^n b_{k-1}^n\right).$$

For $k=0$ only the first and third terms in this sum are left; for $k=n+1$, only the second and fourth.

We compute μ_0 :

$$\mu_0 = \frac{1}{2}(p_n^n + a_n^n) = \frac{1}{2}[(2n+1)+1]a_n^n = 2^n(2n-1)!!.$$

For $k=n+1$ we obtain

$$\mu_{n+1} = \frac{1}{2}(-q_n^n + b_n^n) = 0.$$

Using the obvious equalities

$$b_k^n = kb_{k-1}^n, \qquad a_{n-1+k}^n = (n+k+1)a_{n-k}^n,$$

and the explicit formulas for binomial coefficients, it is not hard to see that $\mu_k = 0$ for all $0 < k < n+1$.

This completes the proof of Theorem 2.

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