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The Maupertuis principle and geodesic flows on the sphere arising from integrable cases in the dynamics of a rigid body

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Introduction

The classical Maupertuis principle, which, incidentally, was exactly 250 years old in 1994, is presented in practically every book devoted to variational calculus in mechanics. At the same time, it should be observed that it has been applied to concrete problems only occassionally. Here we refer, for example, to Novikov's papers [23] and [24], in which it was demonstrated that the Maupertuis principle makes it possible to apply topological methods to find periodic solutions of Hamiltonian systems, in particular, in Kirchhoff's problem. Furthermore, in [24]

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the Maupertuis principle was used in the construction of the theory of many-valued functionals. This theory arises naturally in the study of the motion of a charged particle in a scalar potential field on a smooth manifold in the presence of an effective magnetic field. We also refer to Smolentsev's article [25].

Our paper has a double goal:

1) we describe explicitly (from a modern point of view) the mechanism of the Maupertuis principle using classical integrable dynamical systems as examples;

2) combining this with the new theory concerned with a topological classification of integrable systems (see [1]–[3]), we construct new examples of integrable geodesic flows on the sphere with integrals of degree 3 and 4 and prove that these degrees cannot be reduced.

§1. The general Maupertuis principle

Let M^n be a *compact* smooth Riemannian manifold with metric $g_{ij}(x)$ and let T^*M be the cotangent bundle over M with coordinates x and p, where p is a covector from T_x^*M. We recall that T^*M is a smooth symplectic $2n$-manifold with the standard 2-form $\omega = dp \wedge dx$. We consider the natural mechanical system $v = \operatorname{sgrad} H$ on T^*M, the Hamiltonian H being given by

$$H = g^{ij}(x)p_ip_j + V(x),$$

where g^{ij} is the inverse of the metric tensor and $V(x)$ is a smooth potential on the base space M.

By the well-known *Maupertuis principle*, for sufficiently large h (greater than $\max V(x)$) the integral trajectories of the vector field v coincide with the trajectories of another vector field $\widetilde{v} = \operatorname{sgrad} \widetilde{H}$ on the fixed $(2n-1)$-dimensional isoenergy level $Q^{2n-1} = (H(x,p) = h)$ (which is a smooth manifold), the Hamiltonian $\widetilde{H}$ being given by

$$\widetilde{H} = \frac{g^{ij}(x)}{h - V(x)} p_ip_j.$$

Clearly, $\widetilde{v}$ gives rise to the *geodesic flow* of a Riemannian metric $\widetilde{g}_{ij}$ on T^*M, where

$$\widetilde{g}_{ij} = (h - V(x))g_{ij}(x).$$

Consequently, we can talk of the 'Maupertuis map', transforming the original Hamiltonian vector field v (defined on T^*M) into another vector field $\widetilde{v}$ (defined on the same manifold T^*M). We shall study some important properties of this map.

Theorem 1.

a) *The Hamiltonian field v and its image $\widetilde{v}$ under the Maupertuis map have the same integral trajectories on the fixed energy level $Q^{2n-1} = (H = h)$. It follows that these two Hamiltonian systems are smoothly orbitally equivalent. We recall that here $h > \max V(x)$.*

b) *If v has a smooth integral $f(x,p)$ on the given isoenergy $(2n-1)$-surface Q (we shall refer to such integrals as partial), then $\widetilde{v}$ has also a smooth integral $\widetilde{f}(x,p)$ (no longer partial, but total) on the whole cotangent bundle T^*M (except, perhaps, on the null section $M = \{(x,0)\}$).*

The proof of the theorem follows from the classical proof of the Maupertuis principle. It is easily verified that

$$d\widetilde{H} = \frac{1}{h - V(x)}\, dH$$

on the level surface $Q = (H = h)$.

Also, this implies immediately the formulae for a time substitution along the integral trajectories of v giving the time for $\widetilde{v}$. Namely, if t is the time along the trajectories of v and $\widetilde{t}$ is the time along the trajectories of $\widetilde{v}$, then $d\widetilde{t} = (h - V(x))dt$.

Let us make a comment on the simple but useful assertion b) of Theorem 1. The point is that, by the Maupertuis principle, the integral $\widetilde{f}$ is initially defined only on the isoenergy surface Q. However, it can be extended to the whole cotangent bundle (except, perhaps, the null section) by the natural formula

$$\widetilde{f}(x,p) = f(x, p/|p|),$$

where the norm $|p|$ is understood in the sense of the Riemannian metric $\widetilde{g}_{ij}$.

Here we use the fact that since $\widetilde{v}$ is a geodesic flow, its integral can be extended by homogeneity from one fixed isoenergy surface to the whole space. Some difficulties may arise on the null section, but this will not be our concern, for we shall study Hamiltonian systems on regular $(2n-1)$-dimensional isoenergy manifolds Q (differing from the n-dimensional null section, which is homeomorphic to M).

Remark. The Maupertuis principle can also be used in the case when the Hamiltonian H contains terms linear in the momenta (reflecting, for example, the presence of a magnetic field). See, for example, [23] and [24].

§2. The Maupertuis principle in the dynamics of a massive rigid body

We shall apply Theorem 1 in the important special case when the symplectic 4-manifold T^*M is a smooth submanifold of the six-dimensional linear space $R^6(s_1, s_2, s_3, r_1, r_2, r_3)$, which can be identified with the linear space adjoint to the Lie algebra of the group of motions of the three-dimensional Euclidean space. This Lie algebra is usually denoted by $e(3)$ and then $R^6 = e(3)^*$, the 4-manifold T^*M being defined in R^6 by

$$r_1^2 + r_2^2 + r_3^2 = 1,$$
$$r_1 s_1 + r_2 s_2 + r_3 s_3 = 0.$$

Obviously, these two equations define a four-dimensional manifold diffeomorphic to the cotangent bundle T^*S^2 over the two-dimensional sphere. In various versions of rigid body dynamics r_i and s_i acquire a concrete mechanical meaning. For example, for a rigid body with a fixed point moving in a gravitational force field (that is, for a massive rigid body) r_i are the components of the vertical unit vector (fixed in the three-dimensional space) from the viewpoint of a coordinate system attached to the moving rigid body, while s_i are the components of the kinetic moment vector of the rigid body.

Theorem 2. *Let $v = \operatorname{sgrad} H$ be a Hamiltonian system on the Euclidean space $R^6(s_1, s_2, s_3, r_1, r_2, r_3)$ with Hamiltonian*

$$H(r,s) = \langle B(r)s, s\rangle + V(r),$$

where $\langle\, ,\, \rangle$ is the Euclidean scalar product in R^3, B is a symmetric non-singular positive definite matrix (generally depending on r), and $V(x)$ is a smooth potential. We assume that v is an integrable system on the 4-manifold T^*S^2 (embedded in R^6), the second integral $f(r,s)$ being a smooth function, polynomial of degree n in s, that is, $f(r,s) = \sum_{i=1}^{m} P_i(s)$, where $P_i(s)$ are polynomials of s with coefficients depending on r. We consider the Hamiltonian system $\widetilde{v}$, the image of v under the Maupertuis map.

Then $\widetilde{v}$ (which defines the geodesic flow of some Riemannian metric on S^2) has the second integral $\widetilde{f}$ too (for the construction of which see Theorem 1 above), which is a homogeneous polynomial in s of the same degree as the original integral f.

Proof. By the Maupertuis principle, the new vector field $\widetilde{v}$ is given by a Hamiltonian $\widetilde{H}$ of the form

$$\widetilde{H} = \frac{1}{h - V(r)}\langle B(r)s, s\rangle.$$

First of all, we can assume without loss of generality that either only even powers of s_i or only odd powers of s_i appear in f. Explanation: if even and odd powers of s were both present in f, then, collecting the even and odd powers speparately, one could see immmediately that each of these two groups is an integral by itself. Now we define the new second integral $\widetilde{f}$ of $\widetilde{v}$ by

$$\widetilde{f} = \sum_{i=1}^{m} P_i(s)(\widetilde{H}(s,r))^{(m-i)/2}.$$

This function is a homogeneous polynomial of degree m in s. The point is that, by the above assumption about the degrees of all monomials in f being either odd or even, $(m-i)/2$ is always an integer. It follows that no radicals appear in $\widetilde{f}$.

Now we shall prove that $\widetilde{f}$ is indeed an integral of $\widetilde{v}$. We observe that $\widetilde{f}$ coincides with f on the level set $Q = (H = h)$. It follows that $\widetilde{f}$ is an integral on this energy level. Furthermore, as is easily verified, the homogeneity of $\widetilde{H}$ and $\widetilde{f}$ (in s) implies that the Poisson bracket $\{\widetilde{H}, \widetilde{f}\}$ is also homogeneous (as a polynomial in s). Since this bracket is equal to zero on the given surface Q (on which $\widetilde{f}$ is an integral), $\{\widetilde{H}, \widetilde{f}\}$ is identically equal to zero, that is, $\widetilde{f}$ is an integral on the whole 4-manifold T^*S^2. The theorem has been proved.

§3. The Maupertuis principle and the explicit form of the metric generated on the sphere by a quadratic Hamiltonian on the Lie algebra of the group of motions of R^3

We shall state one more interesting problem. Suppose that, according to the Maupertuis principle, with the original system v on $e(3)^*$ we associate a system $\widetilde{v}$ on the 2-sphere, that is, on its cotangent bundle T^*S^2. However, so far we have described only the meaning of the above correspondence without stating the underlying formulae. In other words, the question is how to find explicit formulae connecting the natural coordinates $(r_1, r_2, r_3, s_1, s_2, s_3)$ with two constraints (imposed upon these six Euclidean coordinates)

$$r_1^2 + r_2^2 + r_3^2 = 1,$$
$$r_1 s_1 + r_2 s_2 + r_3 s_3 = 0$$

on a 4-manifold M^4 with the natural coordinates on the cotangent bundle over the standard 2-sphere? Thus, one must construct four coordinates (x_1, x_2, p_1, p_2) on T^*S^2 from these six coordinates.

We embed the 2-sphere in R^3 with coordinates u_1, u_2, u_3, the sphere being defined in the standard way by $u_1^2 + u_2^2 + u_3^2 = 1$. The other group of coordinates on the cotangent bundle over R^3 will be denoted by v_1, v_2, v_3. These define the momenta. The cotangent bundle over the sphere is thus defined as a 4-manifold in the 6-dimensional space $R^6(u_1, u_2, u_3, v_1, v_2, v_3)$ specified by the two equations

$$u_1^2 + u_2^2 + u_3^2 = 1,$$
$$u_1 v_1 + u_2 v_2 + u_3 v_3 = 0.$$

We identify vectors and covectors with the aid of the *Euclidean scalar product* (the Euclidean metric in R^3).

Now we must define explicitly an embedding μ of the cotangent bundle over the 2-sphere in the linear space $e(3)^*$. It is given by the following explicit formulae:

$$r = u, \qquad s = [u, v],$$

where [,] denotes the vector product in the Euclidean space. In terms of coordinates, this embedding has the form

$$r_i = u_i,$$
$$s_1 = u_2 v_3 - u_3 v_2, \qquad s_2 = u_3 v_1 - u_1 v_3, \qquad s_3 = u_1 v_2 - u_2 v_1.$$

Lemma 1.

a) *Under the above embedding μ of the cotangent bundle T^*S^2 in $e(3)^*$, the image of T^*S^2 coincides with the 4-dimensional orbit of the coadjoint action of the group $E(3)$ of motions of R^3 on the coalgebra $e(3)^*$:*

$$r_1^2 + r_2^2 + r_3^2 = 1,$$
$$r_1 s_1 + r_2 s_2 + r_3 s_3 = 0.$$

b) *Under μ, the canonical symplectic structure on the orbits of the coadjoint representation induces the canonical symplectic structure of the cotangent bundle T^*S^2 on the cotangent bundle over the standard 2-sphere (embedded in the coalgebra).*

Proof. a) It is clear that the equation $u_1^2+u_2^2+u_3^2=1$ turns into $r_1^2+r_2^2+r_3^2=1$ under the given embedding. Furthermore, the vector v orthogonal to the radius vector u turns into the vector product of u and v. It is clear that this product $[u,v]$ is also orthogonal to u. Therefore $s=[u,v]$ satisfies the linear equation

$$r_1s_1+r_2s_2+r_3s_3=0,$$

being the 'image of the orthogonality relation'.

b) Now it remains to verify that the standard Poisson bracket

$$\{u_i,v_j\}=\delta_{ij},\qquad \{u_i,u_j\}=0,\qquad \{v_i,v_j\}=0$$

in $R^6(u,v)$ turns into the standard Poisson–Lie bracket on the coalgebra $R^6(r,s)$:

$$\begin{array}{lll} & \{r_i,r_j\}=0, & \\ \{s_1,s_2\}=s_3, & \{s_1,s_3\}=-s_2, & \{s_2,s_3\}=s_1, \\ \{s_1,r_2\}=r_3, & \{s_1,r_3\}=-r_2, & \{s_2,r_3\}=r_1, \\ \{r_1,s_2\}=-r_3, & \{r_1,s_3\}=r_2, & \{r_2,s_3\}=-r_1. \end{array}$$

This assertion can be verified by direct computation. Therefore we have proved that $T^*R^3(u,v)\to e(3)^*(r,s)$ is a Poisson map. It transforms $R^6(u,v)=T^*R^3(u,v)$ into the 5-dimensional surface defined by $r_1s_1+r_2s_2+r_3s_3=0$ in $R^6(r,s)=e(3)^*$. The map is obviously non-linear, the 'normal bundle' over the sphere being its kernel. The equation $r_1s_1+r_2s_2+r_3s_3=0$ is a special case of $r_1s_1+r_2s_2+r_3s_3=g$, where g is an area constant. Here we take into account that the embedding

$$T^*S^2\to TS^2\to TR^3\to T^*R^3$$

is symplectic. We recall that the Euclidean metric is used to identify the tangent and cotangent bundles.

The lemma has been proved.

Thus, we consider the symplectic embedding (diffeomorphism)

$$\mu\colon T^*S^2(x,p)\to T^*S^2(r,s).$$

Here x and p are the canonical coordinates. To simplify notation we shall write K instead of $\widetilde{H}$.

Lemma 2. *If the Hamiltonian $K(r,s)$ is quadratic in s, that is, it has the form $K=\langle B(r)s,s\rangle$ (where $\langle\,,\,\rangle$ is the Euclidean scalar product), then it is transformed by μ into a Hamiltonian $K(x,p)$ quadratic in p. Consequently, the latter has the form*

$$\sum g^{ij}(x)p_ip_j,$$

where g^{ij} is the inverse tensor to the Riemannian metric $g_{ij}(x)$ on S^2. In particular, the Hamiltonian system with Hamiltonian K turns into the geodesic flow of g_{ij} on the 2-sphere.

Proof. The formulae for μ expressing (r, s) in terms of (u, v) are *linear* in v. The lemma has been proved.

Now we shall write down *explicit formulae* expressing the Riemannian metric g_{ij} in terms of B as well as the *converse formulae* expressing B in terms of g_{ij}.

Theorem 3. *The relationship between B and g_{ij} is as follows. Let B be a given bilinear form. We consider the standard embedding of S^2 into R^3 and define a Riemannian metric $\overline{B}_{\alpha\beta}$ on R^3 by the formulae below. Then the desired Riemannian metric g_{ij} on the sphere will be induced by this metric on R^3. The formulae for $\overline{B}$ read*

$$\overline{B}_{\alpha\beta} = B_{\alpha\beta}/\lambda,$$

with λ being the determinant $\det B(u)|_L$, where $B(u)|_L$ is the restriction of B to the two-dimensional Euclidean plane L orthogonal to the radius vector u, L being equipped with the Euclidean metric, which means that the determinant is computed in the Cartesian coordinates on L.

It follows that $\overline{B}$ is *conformally equivalent* to B. Note that B has a 3×3-matrix, while g_{ij} has a 2×2-matrix. It proves convenient to consider the 'bar' as a map assigning a form to a form. Then the converse of Theorem 3 can be briefly stated as follows.

Theorem 4. *Conversely, B can be reconstructed from the metric $\overline{B}$ in a similar way, namely,*

$$B = \overline{\overline{B}}.$$

We proceed to the proof of Theorems 3 and 4.

Consider the Hamiltonian $K(u, v)$ of a geodesic flow of g_{ij} on S^2. We recall that (u, v) belongs to the cotangent bundle over the sphere. By the definition of K, the value of $K(u, v)$ at (u, v) is the scalar square of v under the metric g^{-1} at a point u on the sphere. Here we consider g^{-1} to be a metric on *vectors*, lowering the indices with the aid of the Euclidean scalar product. On the other hand, from the explicit formulae for the embedding of the cotangent bundle over the sphere into an orbit in the coalgebra we can see that $K(u, v)$ is equal to the scalar square of $[u, v]$ under B. Consequently, taking into account that the action of the vector product in a tangent plane to the sphere can be reduced to a rotation of a vector by $\pi/2$, we obtain the following assertion.

Lemma 3. *Let g be a given form. Then the restriction of B to a tangent plane to the standard 2-sphere can be constructed as follows. To find the scalar product of tangent vectors a and b under B, we must turn each of the vectors through $90°$ and then take their scalar product under g^{-1}.*

On the basis of this lemma, we can now equate the matrices of B and g *on a tangent plane* to the 2-sphere. It is known that there is always an orthonormal basis on the tangent plane in which g can be expressed by a diagonal matrix

$$g = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}.$$

In this basis we find the matrix of B:

$$B = \begin{pmatrix} 1/d & 0 \\ 0 & 1/c \end{pmatrix},$$

which can obviously be rewritten as

$$B = \frac{1}{cd}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \frac{g}{\det g}.$$

Theorem 4 has been proved.

It should be observed that the appearance of $\det g$ can, in fact, be explained by the need to identify vectors with covectors (and vice versa).

From the resulting formula one can see that the 'bar' is an involution in the tangent space to the *standard unit sphere.* (It is not an involution at other points.) This implies Theorem 3.

The appearance of an involution reveals an interesting duality. The 'bar' makes it possible to transform a metric on the sphere (embedded in R^3 in the standard way, but inheriting some metric g of general form from R^3) into another metric. The algorithm of this transformation is, in fact, described in Lemma 3.

Remark. Conversely, on the standard sphere embedded in R^3 we consider the metric induced by a *diagonal* metric on the ambient space R^3. Let this ambient metric have the form

$$ds^2 = I_1(u)\,du_1^2 + I_2(u)\,du_2^2 + I_3(u)\,du_3^2.$$

Then, with the aid of μ^{-1} one can obtain the following Hamiltonian:

$$K = (I_1 I_2 I_3)^{-1} \cdot \frac{I_1(r)s_1^2 + I_2(r)s_2^2 + I_3(r)s_3^2}{I_1^{-1}(r)r_1^2 + I_2^{-1}(r)r_2^2 + I_3^{-1}(r)r_3^2}.$$

In the sense of the duality discovered above, the metric of the ellipsoid is dual to that on the Poisson sphere.

Let us state an interesting problem: to generalize the above construction to the case of the cotangent bundle over a sphere of arbitrary dimension.

§4. Classical cases of integrability in rigid body dynamics and the corresponding geodesic flows on the sphere

The following classical cases of integrability are well known in the theory of a rigid body: the Euler, Lagrange, Kovalevskaya, Goryachev–Chaplygin, and Clebsch cases. All these cases are united by the property that their Hamiltonian is a quadratic form in the momenta plus a potential (that is, no terms linear in the momenta are present). Applying the Maupertuis map, we obtain a variety of corresponding integrable geodesic flows. An interesting question is what flows (and metrics) can be obtained in this way.

We consider the Hamiltonian of the problem of motion of a rigid body. In the general case it has the form

$$H = I_1 s_1^2 + I_2 s_2^2 + I_3 s_3^2 + L(r, s) + V(r),$$

where I_1, I_2, I_3 are constants, L is linear in the momenta, and V is a smooth potential.

Now we shall consider an interesting case when the Hamiltonian H (or the metric g) contains no linear (in the momenta) terms, but includes a potential $V(r)$. What is the corresponding metric on the sphere of radius 1 embedded in the standard way in the Euclidean space $R^3(u_1, u_2, u_3)$?

With the aid of μ, one can construct from it the geodesic flow of the Riemannian metric

$$ds^2 = (h - V(u))(I_1 I_2 I_3)^{-1} \cdot \frac{I_1\, du_1^2 + I_2\, du_2^2 + I_3\, du_3^2}{I_1^{-1} u_1^2 + I_2^{-1} u_2^2 + I_3^{-1} u_3^2}.$$

4.1. The Euler case and the metric on the Poisson sphere. In the Euler case the Hamiltonian has the form

$$H = I_1 s_1^2 + I_2 s_2^2 + I_3 s_3^2.$$

The following metric on the 2-sphere (the so-called *metric on the Poisson sphere*) corresponds to it under the Maupertuis map μ:

$$ds^2 = h(I_1 I_2 I_3)^{-1} \cdot \frac{I_1\, du_1^2 + I_2\, du_2^2 + I_3\, du_3^2}{I_1^{-1} u_1^2 + I_2^{-1} u_2^2 + I_3^{-1} u_3^3},$$

where h is a fixed energy level (that is, $H = h$) in the Euler case. Here it is assumed that the above metric should be restricted to the sphere S^2 embedded in the standard way in $R^3(u_1, u_2, u_3)$. We recall that the same metric on the Poisson sphere can be defined differently. One must consider the rotation group $SO(3)$ equipped with the left-invariant Riemannian metric defined by the diagonal matrix $\mathrm{diag}(I_1, I_2, I_3)$. This matrix gives a scalar product in the Lie algebra of this group. Distributing this scalar product over the whole sphere by left shifts, we obtain a left-invariant metric. One must therefore consider the left action of the circle S^1 on $SO(3)$ and go over to the quotient space of the group under this action, obtaining the 2-sphere as a result. The initial left-invariant metric on the group induces a metric on the base space, that is, on the sphere, which turns out to be the metric on the Poisson sphere.

An interesting question is whether or not the Poisson sphere can be realized as a smooth sphere embedded (or immersed) in R^3?

If the curvature of the metric on the Poisson sphere is positive, then, by the classical Weyl theorem [17], the sphere can be realized as a convex surface in the three-dimensional Euclidean space. It turns out that the curvature is not always positive and, by all appearances, it is generally impossible to embed the Poisson sphere in R^3. For $I_1 = I_2$ this problem was considered by Okuneva in [22], where it was proved that when $(I_1 - I_3)/I_3 < -\frac{1}{3}$ (in which case there are domains of negative curvature on S^2), the Poisson sphere cannot be realized as a surface of revolution in R^3. Apparently, in this case no isometric embedding exists, in general.

4.2. The Lagrange case and the corresponding 'metric of revolution' on the sphere. In the Lagrange case the Hamiltonian has the form

$$H = I_1 s_1^2 + I_1 s_2^2 + I_3 s_3^2 + V(r_3).$$

Here the ellipsoid of inertia is a surface of revolution, since $I_1 = I_2$. The corresponding metric on the sphere has the form

$$ds^2 = (h - V(u_3))(I_1 I_1 I_3)^{-1} \cdot \frac{I_1\, du_1^2 + I_1\, du_2^2 + I_3\, du_3^2}{I_1^{-1} u_1^2 + I_1^{-1} u_2^2 + I_3^{-1} u_3^2}.$$

In other words, the geodesic flow arising from the Lagrange case is a geodesic flow on some sphere of revolution. An interesting problem is to give a geometric description of such spheres of revolution.

4.3. The Clebsch case and the geodesic flow on an ellipsoid. Let us mention an extremely interesting consequence of the Maupertuis principle. Namely, we shall prove that smooth orbital equivalence holds between the Clebsch integrable case and the geodesic flow on an ellipsoid. (We recall that this latter problem is also integrable, which is well known.) This result was obtained by Minkowski and Kozlov by different methods and at different times.

Proposition 1. *Consider an integrable Hamiltonian system v on $e(3) = R^6(s, r)$ describing the motion of a three-dimensional rigid body in an ideal fluid in the classical Clebsch case. In this case the Hamiltonian H has the form*

$$H = \frac{s_1^2}{I_1} + \frac{s_2^2}{I_2} + \frac{s_3^2}{I_3} - (I_1 r_1^2 + I_2 r_2^2 + I_3 r_3^2).$$

The second integral f of this system has the form

$$f = s_1^2 + s_2^2 + s_3^2 + (I_2 I_3 r_1^2 + I_3 I_1 r_2^2 + I_1 I_2 r_3^2).$$

Then v restricted to the three-dimensional energy level $H = 0$ is smoothly orbitally equivalent to the geodesic flow $\widetilde{v}$ of the standard Riemannian metric on the ellipsoid in the three-dimensional Euclidean space defined by

$$I_1 x^2 + I_2 y^2 + I_3 z^2 = 1$$

in the Cartesian coordinates x, y, z.

Proof. We consider the three-dimensional level set $Q = (H = 0)$ (that is, $h = 0$) and apply the general scheme of the Maupertuis map presented above. Then the trajectories of v coincide with those of the system $\widetilde{v}$ with Hamiltonian $\widetilde{H}$ of the form

$$\widetilde{H} = \left(\frac{s_1^2}{I_1} + \frac{s_2^2}{I_2} + \frac{s_3^2}{I_3}\right) \Big/ (I_1 r_1^2 + I_2 r_2^2 + I_3 r_3^2).$$

The Hamiltonian $\widetilde{H}$ gives rise to a Riemannian metric on the 2-sphere. It is easily verified that this metric coincides with the metric of the ellipsoid defined by the equation in the theorem. The proposition has been proved.

Here the second integral of $\widetilde{v}$ has the form

$$\widetilde{f} = s_1^2 + s_2^2 + s_3^2 + (I_2 I_3 r_1^2 + I_3 I_1 r_2^2 + I_1 I_2 r_3^2)\frac{s_1^2/I_1 + s_2^2/I_2 + s_3^2/I_3}{I_1 r_1^2 + I_2 r_2^2 + I_3 r_3^2}.$$

Remark. Consider another energy level $H = h$, where h is now different from zero. Following the same scheme, with the aid of the Maupertuis map we obtain a new Hamiltonian system $\widetilde{v}$, the geodesic flow of a Riemannian metric on the two-dimensional sphere. This system has the following Hamiltonian:

$$\widetilde{H} = \left(\frac{s_1^2}{I_1} + \frac{s_2^2}{I_2} + \frac{s_3^2}{I_3}\right) \Big/ \left(I_1 r_1^2 + I_2 r_2^2 + I_3 r_3^2 + h\right).$$

Again, this system is obviously integrable. Its second integral has the form

$$\widetilde{f} = s_1^2 + s_2^2 + s_3^2 + \left(I_2 I_3 r_1^2 + I_3 I_1 r_2^2 + I_1 I_2 r_3^2\right)\frac{s_1^2/I_1 + s_2^2/I_2 + s_3^2/I_3}{I_1 r_1^2 + I_2 r_2^2 + I_3 r_3^2 + h}.$$

Therefore we can see that the metric of the ellipsoid belongs to a one-parameter family of Riemannian metrics (being no longer ellipsoid metrics), whose geodesic flows are nevertheless integrable. It would be extremely interesting to find out what Riemannian metrics can thereby be obtained on the two-dimensional sphere. Since they are integrable with the aid of an integral quadratic in the momenta, it follows by the classical theorem that they are *Birkhoff* metrics, that is, they can be written in the well-known form (discovered by Birkhoff and then studied by Kolokol'tsov) for a suitable choice of conformal coordinates on the sphere. However, it is quite difficult to find 'suitable conformal coordinates' (the algorithm for finding such coordinates being unclear itself). Therefore, the problem of desciribing the geometric properties of such metrics remains topical.

4.4. The Goryachev–Chaplygin case and the corresponding integrable geodesic flow on the sphere. The cubic integral cannot be reduced to a quadratic or linear one. Here we present in detail the results by Bolsinov and Fomenko briefly published in [16].

We apply the same formula for the Goryachev–Chaplygin integrable case. The Goryachev–Chaplygin Hamiltonian has the form

$$H = s_1^2 + s_2^2 + 4s_3^2 + r_1.$$

The Goryachev–Chaplygin integral is given by

$$f = s_3(s_1^2 + s_2^2) - (r_3 s_1)/2.$$

By the Maupertuis principle, we construct from H the following new Hamiltonian $\widetilde{H}$ on the cotangent bundle over the sphere:

$$\widetilde{H} = \frac{s_1^2 + s_2^2 + 4s_3^2}{h - r_1},$$

where h is a constant greater than 1. Then the integral f turns into an integral $\widetilde{f}$ of the form

$$\widetilde{f} = s_3(s_1^2 + s_2^2) - \frac{r_3 s_1}{2(h - r_1)}(s_1^2 + s_2^2 + 4s_3^2).$$

The Riemannian metric of the corresponding geodesic flow on the sphere reads

$$ds^2 = \frac{h - u_1}{4} \frac{du_1^2 + du_2^2 + 4du_3^2}{u_1^2 + u_2^2 + u_3^2/4}.$$

To write down the integral of the corresponding geodesic flow, it suffices to replace r and s in the formula for $\widetilde{f}$ by their expressions in terms of u and v given in [15]. We shall not do this in view of the complexity of the resulting expressions.

Theorem 5 (Bolsinov and Fomenko [16]). *The Goryachev–Chaplygin integrable case generates a Riemannian metric on the sphere having a Bott geodesic flow integrable with the aid of the above-mentioned integral of degree* 3. *This integral cannot be reduced to a quadratic one.*

Proof. We consider the rough molecule W of the given geodesic flow of the Goryachev–Chaplygin metric on the sphere. As is known, this flow is continuously orbitally equivalent to the original dynamical system of rigid body dynamics (the Goryachev–Chaplygin case). The rough molecule W of the latter integrable system has already been computed by Oshemkov in [12]. It has the form presented in Fig. 1. To continue our proof we assume the contrary, that is, that the Goryachev–Chaplygin integral on the sphere can be reduced to a quadratic one. But in this case we could use Nguyen and Polyakova's paper [5], in which they gave a complete computation of the molecules W^* of all geodesic flows on the sphere integrable with the aid of *quadratic* and *linear integrals*. All these molecules are listed explicitly.

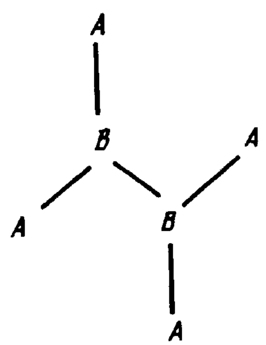

Figure 1

Using this result, we can see that the molecule of the Goryachev–Chaplygin flow on the sphere would have to have one of the two forms presented in Fig. 2. The molecule in Fig. 2a would correspond to the case when the Goryachev–Chaplygin integral could be reduced to a quadratic one, while that in Fig. 2b would correspond to the case when it could be redued to a linear one. Here W_1 is a tree, all of whose branches are *directed upwards*, while all the branches of W_2 are *directed downwards*. Furthermore, the atoms forming the vertices of W_1 and W_2 must have a special form. In particular, they must not contain any 'stars' (see [5]). Comparing these graphs with that in Fig. 1, we can see that the latter ***has no such structure***. Since W is a rough topological invariant of an integrable system, we obtain a contradiction. The proof is completed.

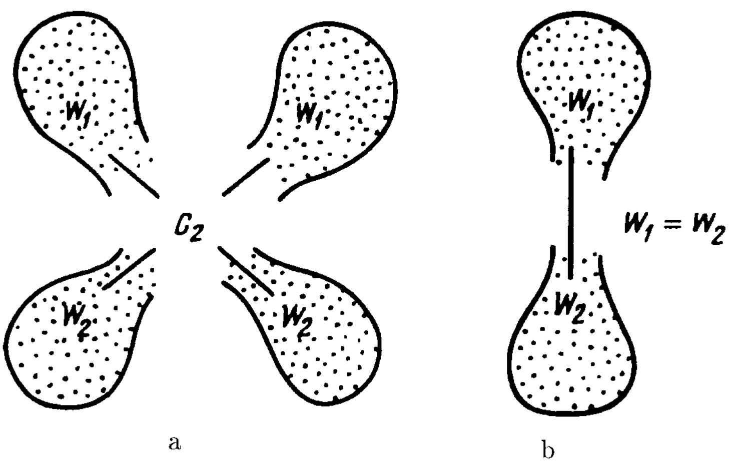

Figure 2

4.5. The Kovalevskaya case and the corresponding integrable geodesic flow on the sphere. The integral of degree four cannot be reduced to a quadratic one. Here we present in detail Bolsinov and Fomenko's results briefly published in [16].

The Kovalevskaya Hamiltonian has the form

$$H = \frac{1}{2}(s_1^2 + s_2^2 + 2s_3^2) + r_1.$$

The Kovalevskaya integral is given by

$$f = \left(\frac{s_1^2}{2} - \frac{s_2^2}{2} - r_1\right)^2 + (s_1 s_2 - r_2)^2.$$

By the Maupertuis principle, we can construct from H the following new Hamiltonian $\widetilde{H}$ on the cotangent bundle over the sphere:

$$\widetilde{H} = \frac{s_1^2 + s_2^2 + 2s_3^2}{h - r_1},$$

where h is a constant greater than 1. Then f turns into an integral $\widetilde{f}$ of the form

$$\widetilde{f}=\left(\frac{s_1^2}{2}-\frac{s_2^2}{2}-r_1\frac{s_1^2+s_2^2+2s_3^2}{h-r_1}\right)^2+\left(s_1s_2-r_2\frac{s_1^2+s_2^2+2s_3^2}{h-r_1}\right)^2.$$

The Riemannian metric of the corresponding geodesic flow on the sphere has the form

$$ds^2=\frac{h-u_1}{2}\frac{du_1^2+du_2^2+2du_3^2}{u_1^2+u_2^2+u_3^2/2}.$$

To write down the integral of the corresponding geodesic flow, it suffices to replace r and s in the formula for $\widetilde{f}$ by their expressions in terms of u and v given in [15]. We shall not do this because of the complexity of the resulting expressions.

Theorem 6 (Bolsinov and Fomenko [16]). *The Kovalevskaya integrable case generates a Riemannian metric on the sphere having a Bott geodesic flow integrable with the aid of the above-mentioned integral of degree* 4. *This integral cannot be reduced to a quadratic one.*

Proof. Consider the molecule W of the given geodesic flow of the Kovalevskaya metric on the sphere. As is known, this flow is continuously orbitally equivalent to the original dynamical system of rigid body dynamics (the Kovalevskaya case). The molecule W of the latter integrable system was computed before by Oshemkov in [12] (see also [11]). It has the form presented in Fig. 3. To continue the proof we assume the contrary, that is, that Kovalevskaya's integral on the sphere can be reduced to a quadratic one. In this case we can use Nguyen and Polyakova's article [5] again, in which the authors present a complete computation of the molecules W^* of all geodesic flows on the sphere integrable with the aid of *quadratic* and *linear integrals*. All these molecules are listed explicitly.

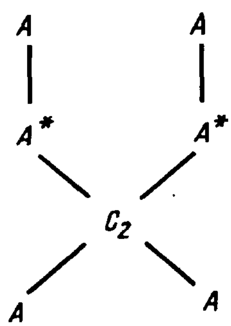

Figure 3

Using this result, we can see that the molecule of the Kovalevskaya flow on the sphere would have to have one of the two forms in Fig. 2. The molecule in Fig. 2a would correspond to the case when the Kovalevskaya integral could be reduced to a quadratic one, while that in Fig. 2b to the case when it could be reduced to a linear one. As can be seen in Fig. 3, in the Kovalevskaya case the molecule W contains two atoms A^*, neither of which appears in any of the molecules computed by Nguyen and Polyakova. This contradiction proves the theorem.

§5. Integrable metrics on the torus and on the sphere

We observe that the Goryachev–Chaplygin and Kovalevskaya integrable cases generate, in fact, a whole (at least *one-parameter*) family of Riemannian metrics on the sphere. As can be seen from the formulae, the coefficients of the metric contain a parameter h, which can be varied arbitrarily. We thus obtain two one-parameter families of integrable geodesic flows of metrics with integral of degree 3 and 4, respectively, which cannot be reduced to quadratic ones.

It follows that there are metrics on the sphere whose geodesic flows are integrable with the aid of integrals of degree 1, 2, 3, 4. We refer to such metrics as *integrable 1-metrics, 2-metrics, 3-metrics*, and *4-metrics* (and the set of these metrics we call *1–2–3–4-metrics*).

Integrable 1-metrics and 2-metrics have been described completely. To begin with, we consider the local aspect of this problem. Let x_1, x_2 be local conformal (isothermal) coordinates of a Riemannian metric, and let p_1 and p_2 be the associated canonical momenta. In terms of these variables the Hamiltonian of the geodesic flow takes the form

$$H = \frac{p_1^2 + p_2^2}{2\Lambda(x_1, x_2)}.$$

Theorem 7 (Birkhoff [9]). *If there is an integral f linear in the momenta, then $f = p_1$ and Λ is independent of x_1 in some conformal coordinates. But if an integral quadratic in the momenta and independent of H exsits, then $\Lambda = \lambda(x_1) + \mu(x_2)$ in some coordinates.*

The coordinate x_1, of which H is independent, is called *cyclic* and the corresponding momentum p_1 is called a *cyclic integral.* Thus, the presence of linear integrals is connected with the existence of 'hidden' cyclic variables.

The Hamiltonian of a geodesic flow with a quadratic integral can be reduced to the form

$$H = \frac{p_1^2 + p_2^2}{2(\lambda(x_1) + \mu(x_2))}.$$

Systems of this form are called *Liouville systems.* The variables x_1, p_1 and x_2, p_2 can be separated: the equations admit two quadratic integrals

$$f_1 = \frac{p_1^2}{2} - H\lambda(x_1), \qquad f_2 = \frac{p_2^2}{2} - H\mu(x_2).$$

Thus, the presence of an additional quadratic integral implies the existence of 'hidden' separated variables.

In fact, Birkhoff considered a more general problem concerning *conditional* polynomial integrals of degree one and two in the momenta (these are integrals on a single level set of the energy integral). Birkhoff's theorem is also valid in this case, except that a suitable time substitution must be made in addition.

Now we consider the global problem of constructing 1-metrics and 2-metrics. First of all, we observe that a closed surface M of genus > 1 admits no 1- or 2-metrics at all. This is a consequence of a more general result on the absence of non-trivial integrals of a geodesic flow that are analytic in the momenta and, in particular, integrals polynomial in the momenta [18]. Therefore, it remains to consider the cases when M is homeomorphic to the torus T^2 or sphere S^2. As is well known, on the torus one can always introduce global conformal coordinates. This means that the Hamiltonian of the geodesic flow on the torus can be assumed to have the form $H = \dfrac{p_1^2 + p_2^2}{2\Lambda(x_1, x_2)}$, where Λ is a positive 2π-periodic function of x_1 and x_2.

Theorem 8 (Birkhoff, Kolokol'tsov, Babenko, Nekhoroshev). *If there is an integral linear in the momenta, then*

$$\Lambda = \lambda(mx_1 + nx_2)$$

in some angular conformal coordinates, $\lambda(\cdot)$ being a 2π-periodic function and m, n being integers. But if an additional quadratic integral exists, then

$$\Lambda = \lambda(mx_1 + nx_2) + \mu(-nx_1 + mx_2)$$

in some conformal coordinates, $\lambda(\cdot)$ and $\mu(\cdot)$ being 2π-periodic functions and m, n integers.

We point out that a linear integral is now always generated by an angular cyclic coordinate. Here is a simple example: $\Lambda \equiv 1$ and $f = p_1 + \sqrt{2}p_2$. In fact, this theorem appears in [10], although it contains an error discovered by Babenko and Nekhoroshev. It asserts that with the aid of a suitable unimodular linear transformation one can set $m = 1$ and $n = 0$ in the expressions for Λ. It turns out that either conformality or unimodularity must then be violated. A more general theorem on conditionally linear and conditionally quadratic (in the sense of Birkhoff) integrals of a geodesic flow on the torus was obtained in [19].

The description of integrable 1-metrics and 2-metrics on the two-dimensional sphere represents a more complex problem.

Theorem 9 (Kolokol'tsov [10]). *If a geodesic flow on the sphere has a linear integral, then Λ has the form $\Lambda = \lambda(x_1^2 + x_2^2)$ in some conformal coordinates x_1, x_2 on the sphere with one point pricked out, $\lambda(\cdot)$ being a smooth function such that $\lambda(\xi) = (c + o(1))/\xi$ as $\xi \to \infty$.*

If an additional quadratic integral exists, then in some conformal cordinates Λ has the form

$$\Lambda = \frac{\lambda(u(x_1, x_2)) + \mu(v(x_1, x_2))}{|4z^2 + g_2 z + g_3|}, \qquad z = x_1 + ix_2,$$

where $g_2^3 - 27g_3^2 \neq 0$, u and v being the real and imaginary parts of the transformation $w = \mathcal{P}^{-1}(z)$, where $\mathcal{P}(z|g_2, g_3)$ is the Weierstrass function with a pair of periods of the form $\omega_1, i\omega_2$ with real ω_1, ω_2, and where λ and μ are smooth functions such that

a)
$$\lambda(u) = (u - k\omega_1/2)^2(c + o(1)) \quad \textit{as} \quad u \to k\omega_1/2,$$
$$\mu(v) = (v - k\omega_2/2)^2(c + o(1)) \quad \textit{as} \quad v \to k\omega_2/2,$$

for any integer k and any $c > 0$,

b) *the values of* λ *and* μ *on the intervals* $[\omega_1/2, \omega_1]$ *and* $[\omega_2/2, \omega_2]$ *are determined by their values on* $[0, \omega_1/2]$ *and* $[0, \omega_2/2]$ *according to the formulae*

$$\lambda(\omega_1/2 + \tau) = \lambda(\omega_1/2 - \tau), \qquad \tau \in [0, \omega_1/2],$$
$$\mu(\omega_2/2 + \tau) = \mu(\omega_2/2 - \tau), \qquad \tau \in [0, \omega_2/2],$$

c) λ *and* μ *are periodic with periods* ω_1 *and* ω_2 *respectively.* (*Then* $\lambda(u(z)) + \mu(v(z))$ *is independent of the choice of the value of the many-valued function* $\mathcal{P}^{-1}(z)$.)

As opposed to these metrics, integrable 3-metrics and 4-metrics have not been described completely. So far we have presented only some separate initial examples of such metrics, although we are convinced that these one-parameter metrics can, in fact, be included in some natural *many-parameter* (or even functional) families of metrics integrable with the aid of integrals of degree 3 or 4.

We say that a metric is *non-singularly integrable* if its geodesic flow can be integrated with the aid of some Bott integral (on a non-degenerate isoenergy level).

§6. Conjectures

(a) On the torus there are no other non-singularly integrable metrics, except for the well-known integrable 1–2-metrics.
(b) On the sphere any non-singularly integrable metric coincides with one of the integrable 1–2–3–4-metrics.
(c) A counter conjecture for the case of the sphere.

Conjectures (a) and (b) were stated by Kozlov and Fomenko. In their formulation the *non-singularity* condition for the metric can possibly be relaxed to be replaced just by *smoothness.*

Problem. *Is it true that any non-singularly integrable* (*smooth*) *Riemannian metric on the torus must be an integrable* 1–2-*metric* (*all such metrics are known*) *and on the sphere it must coincide with some metric of type* 1–2–3–4 (*at this moment only* 1–2-*metrics have been completely described*), *that is, must have an integral of degree* 1, 2, 3, *or* 4?

In other words, it is claimed that if the geodesic flow of some smooth metric on the torus (respectively, on the sphere) is integrable with the aid of a smooth non-singular integral (or one polynomial in the momenta), then the flow must have an integral of degree 1 or 2 in the case of the torus and not exceeding 4 in the case of the sphere.

One of the authors of the present article, namely A. V. Bolsinov, has expressed *his own* opposite opinion (in the case of the sphere), which is the following:

In the case of the sphere there are no topological obstacles to the existence of geodesic flows integrable with the aid of an integral of arbitrary degree.

Therefore Bolsinov states a counter conjecture: *on the sphere there are integrable geodesic flows of metrics whose integrals have arbitrarily high degree* (*in the momenta*) *and cannot be reduced to an integral of degree* 1, 2, 3, *or* 4.

This assertion is supported by the following local result.

Theorem 10 (Kozlov). *There are systems with Hamiltonian*

$$H = \frac{p_1^2 + p_2^2}{2\Lambda(x_1, x_2)}$$

that admit in the domain $D_x \times R_p^2$ (D being a disc on the x_1, x_2 plane) a polynomial integral of any given degree n in the momenta and independent of H, and do not admit any polynomial integral of degree $< n$ independent of H.

For the time being, we can satisfactorily substantiate neither the first nor the second conjecture. We shall therefore proceed to the case of the torus (for which there is no difference of opinion between the authors).

Theorem 11 (Kozlov and Denisova [20]). *We assume that Λ is a trigonometric polynomial and that the geodesic flow on the torus has a polynomial integral independent of H. Then there is an additional polynomial integral of degree $\leqslant 2$ in the momenta.*

According to the Weierstrass approximation theorem, any metric on the torus can be approximated as closely as required by such metrics. The above conjecture is therefore proved for an everywhere dense set of Riemannian metrics on the torus.

In the general case when Λ is an arbitrary smooth function on the torus, a weaker result can be obtained in this direction.

Theorem 12 (Kozlov and Denisova [21]). *We assume that the geodesic flow on the torus has an additional polynomial integral f of degree n such that*

a) *if n is even, then f is an even function of p_1 and p_2,*
b) *if n is odd, then f is an even function of p_1 (or p_2) and an odd function of p_2 (or p_1).*

Then there is a polynomial integral of degree $\leqslant 2$ independent of H.

The problem of existence of natural integrable mechanical systems on T^2 with integrals of higher degree in the momenta was partially studied by Byalyi in [13], and Kozlov and Treshchev in [14]. For example, in [13] Byalyi considered the Hamiltonian system $v = \operatorname{sgrad} H$ with Hamiltonian

$$H = (p_1^2 + p_2^2)/2 + V(x, y),$$

where V is a 2π-periodic function of x and y. The system $v = \operatorname{sgrad} H$ determines the motion of a material point on the two-dimensional torus in a potential field. A general problem is to find a potential V such that there exists an integral F_n polynomial of degree n in the momenta with 2π-periodic coefficients. Since the odd and even parts of F_n taken separately are then first integrals, F_n can be assumed to be the sum of homogeneous polynomials of odd or even degrees only. The cases of linear and quadratic integrals are among the classical ones (see Birkhoff [9], Kolokol'tsov [10], and Byalyi [13]). It is known that

(1) an integral F_1 linear in the momenta exists if and only if $V = f(mx + ny)$, where m, n are integers and f is a 2π-periodic function; then $F_1 = mp_2 - np_1$;

(2) an integral F_2 quadratic in the momenta exists if and only if $V = f_1(m_1x+n_1y)+f_2(m_2x+n_2y)$, where m_i and n_i are integers and $m_1m_2/n_1n_2 = -1$, and where f_i are 2π-periodic; then $F_2 = (x_1 + x_2)p_1^2 + 4p_1p_2 - (x_1 + x_2)p_2^2 + 2(x_1 - x_2)(f_1 + f_2)$ and $x_i = m_i/n_i$.

For higher degrees, let us mention the following results due to Byalyi.

Proposition 2 (Byalyi [13]). *An integral F_3 cubic in the momenta exists for a Hamiltonian H on T^2 if and only if case (1) is realized, that is, there exists an integral F_1 linear in the momenta and, this being the case, F_3 can be explicitly expressed in terms of F_1 and H.*

Proposition 3 (Byalyi [13]). *An integral F_4 of degree four in the momenta exists for a Hamiltonian H on T^2 if and only if case (2) is realized, that is, there exists an integral F_2 quadratic in the momenta and, this being the case, F_4 can be explicitly expressed in terms of F_2 and H.*

The proof of either of these theorems rests on a detailed study of the equation $\{F_n, H\} = 0$. For $n > 4$ the computations become quite cumbersome, even though they probably lead to the same results.

Kozlov and Treshchev [14] considered this problem from another viewpoint: the potential $V(x_1, \dots, x_n)$ of the natural system with Hamiltonian

$$H = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} p_i p_j + V(x_1, \dots, x_n)$$

is assumed to be a trigonometric polynomial of $x_1, \dots, x_n$ on T^n, but there are no restrictions on the degree of additional integrals. The basic theorem giving a necessary and sufficient condition to be satisfied by $V(x_1, \dots, x_n)$ in order to ensure complete integrability (in the sense of Birkhoff) of a Hamiltonian system with positive definite quadratic form $\frac{1}{2} \sum_{i,j=1}^{n} a_{ij} p_i p_j$ has the following important consequence. If a Hamiltonian system with Hamiltonian

$$H = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} p_i p_j + V(x_1, \dots, x_n)$$

has n independent polynomial integrals with independent leading homogeneous forms, then there are n independent involutory polynomial integrals of degree not exceeding 2.

Thus, the results of [13], [14], [20], and [21] provide some convincing evidence that no natural integrable mechanical systems with non-trivial integrals of higher degree in the momenta should be expected to exist on T^2. Nevertheless, the problem remains open for the time being.

§7. The complexity of integrable geodesic flows of 1–2-metrics on the sphere and on the torus

Selivanova [4] and Nguyen and Polyakova [5] obtained a topological classification of integrable geodesic flows on T^2 and S^2 having additional integrals f linear or quadratic in the momenta. Later on, this enabled Selivanova, Polyakova, and Nguyen to classify the complexity of these integrable geodesic flows on the sphere and on the torus (see [4], [5], and below).

Theorem 13 (Nguyen and Polyakova). *Integrable geodesic flows (Riemannian metrics) on the sphere having additional integrals linear or quadratic in the momenta fill the domain $A \cup B$ in the molecular complexity table* (Fig. 4). *Here*

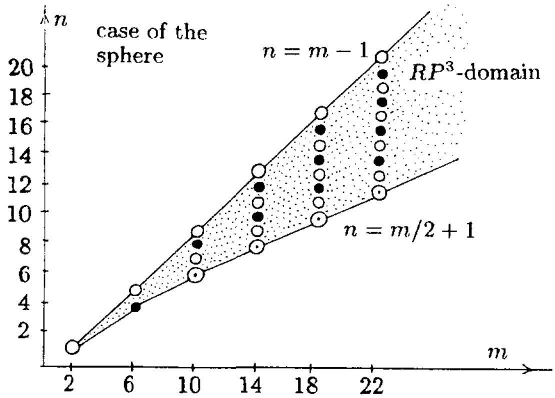

Figure 4

a)

$$A = \Big\{(m,n) = (6,4) \text{ or } \frac{m}{2} + 3 \leqslant n \leqslant m - 2, \ m \geqslant 6,$$

$$\text{where } m = 4k+2, \ n = 2l \text{ and } k,l \text{ are arbitrary non-negative integers} \Big\}.$$

This set A, which in the table represents integrable geodesic flows with a quadratic additional integral, is designated by black discs in Fig. 4.

b)

$$B = \Big\{(m,n) = (2,1) \text{ or } \frac{m}{2} + 2 \leqslant n \leqslant m - 1, \ m \geqslant 6,$$

$$\text{where } m = 4k+2, \ n = 2l+1 \text{ and } k,l \text{ are arbitrary non-negative integers}\Big\}.$$

This set B, which in the table represents integrable geodesic flows with a linear additional integral, is designated by white discs in Fig. 4.

Theorem 14 (Selivanova). *Integrable geodesic flows on the torus fill the domain C in the molecular complexity table* (Fig. 5). *Here*

$$C = \Big\{(m,n) = (0,1) \text{ or } \frac{m}{2} + 2 \leqslant n \leqslant m, \ m = 4k, \ n = 2(l+1),$$

$$\text{where } k,l \text{ are arbitrary non-negative integers}\Big\}.$$

In Fig. 5 *the points of this set are shown as black discs.*

In the case of geodesic flows on the torus the isoenergy 3-surface $Q^3 = \{H = \text{const}\}$ is diffeomorphic to the torus T^3. In the case of the sphere it is diffeomorphic to the projective space, that is, $Q^3 = RP^3$. All 'mathematically existing' integrable Hamiltonian systems (of general type; see above) on the isoenergy 3-sphere have

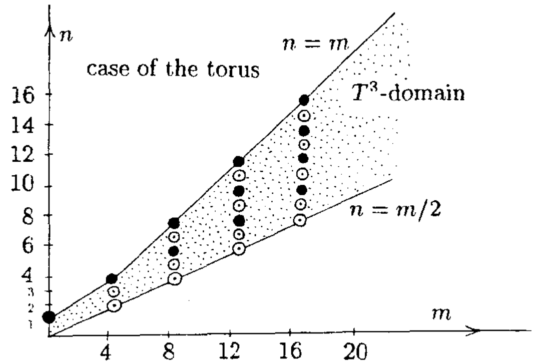

Figure 5

been classiffied by Nguyen and Fomenko [6]. The complexities of all 'mathematically existing' integrable systems on T^3 and on the projective space RP^3 were described by Nguyen [7], who computed the shape of the domain in the complexity table filled with the complexities of such integrable systems. These domains have been called the S^3-domain, the T^3-domain, and the RP^3-domain, respectively [1], [6], [7].

As we can see in Figs. 4 and 5, the zones representing the complexities of integrable geodesic flows of 1–2-metrics on the sphere (and, respectively, on the torus) form a 'net' (a 'net subset') inside the RP^3-domain (respectively, the T^3-domain).

§8. A rougher conjecture: the complexities of non-singularly integrable metrics on the sphere or on the torus coincide with those of the known integrable 1–2-metrics

This conjecture has been stated by Fomenko. It is based on some 'experimental' facts.

Conjecture on the complexity of integrable metrics. *Let g_{ij} be an arbitrary non-singularly integrable Riemannian metric on the torus (or sphere) (that is, the corresponding geodesic flow is integrable in the sense of Liouville). Then in the case of the torus the complexity of this metric (its geodesic flow) is the same as that of some integrable 1–2-metric (all such complexities have already been computed), and in the case of the sphere it is equal to that of some integrable 1–2–3–4-metric (the complexities of which have been described only partially so far).*

As we have proved above, on the sphere there are integrable metrics with an additional integral of degree 3 or 4 irreducible to a quadratic one, for example, the Goryachev–Chaplygin metric and the Kovalevskaya metric. It is easily seen that the complexity of the Goryachev–Chaplygin metric is equal to (6, 5), that is, it lies in the domain filled with the complexities of 1–2-metrics. At the same time, the complexity (8, 6) of the Kovalevskaya metric clearly *does not belong* to this zone (Fig. 4). We also observe that Kovalevskaya's molecule contains two atoms of the form A^* (while the geodesic flows of 1–2-metrics have no such atoms).

If the general conjecture stated above is true, then the point representing any non-singularly integrable metric *on the torus* must lie in the remarkable domain in the complexity table discovered by Selivanova. This domain is designated by black discs in Fig. 5. The form of the analogous domain for the sphere remains unclear for the time being: so far only the complexities of 1–2-metrics have been described, but it is entirely unclear what the complexities of 3–4-metrics look like.

An integrable Riemannian metric on a two-dimensional manifold M^2 is called *orientable* if all saddle critical circles of the corresponding geodesic flow (= Hamiltonian system) have orientable separatrix diagrams.

It turns out that in the class of orientable integrable metrics on the sphere Fomenko's conjecture on their complexity is 'almost true'. Namely, the following assertion holds.

Proposition 4 (Nguyen, Polyakova, and Kalashnikov (jr)). *The number m of critical circles of an arbitrary orientable non-singularly integrable Riemannian metric on the sphere has the form $m = 4k + 2$ for some integer k.*

Corollary 1 (Nguyen and Polyakova). *Orientable non-singularly integrable geodesic flows on the sphere lie inside the following 'net subset' in the RP^3-domain represented in Fig. 4 by white and black discs as well as white discs containing a dot. Here the geodesic flows with linear and quadratic integrals are represented by white and black discs, respectively. White discs containing a dot stand for presently unknown (and maybe not existing at all) integrable Riemannian metrics.*

Therefore, as we can see, the 'complexity zone of all integrable orientable metrics' practically coincides with the 'zone of linearly or quadratically integrable metrics', except for the straight line along the lower boundary of the angle in Fig. 4. An extremely interesting problem is to complete the study of this 'special straight line' on the boundary (and thus to provide a final answer to the question stated above).

The assertion that $m = 4k$ (analogous to the theorem above) holds for an arbitrary non-singularly integrable geodesic flow of a Riemannian metric on the torus. Here n is an even number (see above). Therefore, the '1–2-integrability zone' (presented in the table of complexities of the classical Liouville integrable 1–2-metrics on the torus) covers at least 'a half' of the domain corresponding to all 'mathematically feasible' integrable geodesic flows (of non-singular orientable metrics) on the torus. In Fig. 5 black discs represent the Liouville geodesic flows on the torus and white discs with a dot stand for the presently unknown (and maybe not existing at all) integrable geodesic flows on the torus.

§9. The geodesic flow on an ellipsoid is topologically orbitally equivalent to the Euler integral case in the dynamics of a rigid body

Apart from the Maupertuis principle, there are other methods of establishing an isomorphism between various dynamical systems.

In this section we mention one such method developed recently by Bolsinov and Fomenko. For some other methods and examples of isomorphisms see, for example, [30]–[32].

As is well known, the geodesic flow of the metric induced on an ellipsoid in the Euclidean space R^3 is a completely integrable Hamiltonian system [26] (Jacobi).

We also consider another known integrable system, namely, the so-called Euler case in the dynamics of a rigid body in R^3. We prove that *these two dynamical systems are topologically orbitally equivalent.*

Definition. Two smooth dynamical systems are said to be topologically (smoothly) orbitally equivalent if there is a *homeomorphism* (*diffeomorphism*) from one manifold to the other that transforms the trajectories of the first system into those of the other system and preserves their orientation. (It is not required that the time along the trajectories should be preserved.)

The problem of orbital equivalence of dynamical systems has been discussed in many papers.

Since the three-dimensional level surfaces of an integrable Hamiltonian with two degrees of freedom are invariant under the flow, it suffices to study the orbital classification of integrable systems (with two degrees of freedom) on three-dimensional energy levels. In what follows, when talking of a three-axial ellipsoid we shall always assume that it is not a surface of revolution. Similarly, in the case of a rigid body we shall assume that its moments of inertia are distinct.

Theorem 15 (Bolsinov and Fomenko [15]). *The Euler integrable case with zero constant area is topologically orbitally equivalent to the integrable geodesic flow on a three-axial ellipsoid* (*the Jacobi problem*).

More explicitly, this means the following.

a) The dynamical system describing the Euler integrable case with zero constant area (on a three-dimensional energy level and generally on the whole four-dimensional symplectic manifold M^4, the cotangent bundle over the 2-sphere with zero section removed) is topologically orbitally equivalent to the geodesic flow on an ellipsoid in the three-dimensional space (the Jacobi problem) (respectively, on a three-dimensional energy level and generally on the whole cotangent bundle over the 2-sphere with the zero section removed).

b) This means that for every rigid body in the Euler case with a fixed ellipsoid of inertia there is a unique (to within similarity) ellipsoid in the three-dimensional Euclidean space whose geodesic flow is topologically orbitally equivalent to the given system of Euler equations, that is, the dynamical system describing the free rotation of a rigid body about its centre of mass.

c) Conversely, for the geodesic flow of an arbitrary ellipsoid in R^3 there is a rigid body revolving freely about its centre of mass, whose ellipsoid of inertia is uniquely defined (to within similarity), such that the dynamical system of Euler equations describing its motion is topologically orbitally equivalent to the original geodesic flow of the metric on the ellipsoid in R^3.

d) The explicit formulae connecting the squared half-axes a, b, c of the Jacobi ellipsoid and the principal moments of inertia $1/A, 1/B, 1/C$ of the rigid body in the Euler case orbitally equivalent to it can be obtained from the equalities

$$-\frac{\int_{-b}^{-a}\Phi(u,c)\,du}{\int_0^{+\infty}\Phi(u,c)\,du}-1=-\frac{C}{\sqrt{(C-A)(C-B)}},$$
$$\frac{\int_{-c}^{-b}\Phi(u,a)\,du}{\int_0^{+\infty}\Phi(u,a)\,du}-1=-\frac{A}{\sqrt{(C-A)(B-A)}},$$

where $\Phi(u,t)=\sqrt{\dfrac{u}{(u+a)(u+b)(u+c)(u+t)}}$.

Remark. In the classical theory of a real rigid body the principal moments of inertia satisfy the triangle inequalities. However, the Euler–Poisson equations make sense for arbitrary moments of inertia. Therefore, instead of restricting the 'set of rigid bodies' by the triangle inequalities, we shall consider all possible triples $1/A, 1/B, 1/C$ that satisfy just one condition $1/A > 1/B > 1/C$. The same Euler–Poisson equations describe the geodesics on the Poisson sphere and are valid for any $1/A > 1/B > 1/C$.

These results follow from the general theory of orbital classification of non-singular integrable systems with two degrees of freedom on three-dimensional energy levels developed by Bolsinov and Fomenko in [28] and [29].

A system is called *non-singular* if it has a *Bott integral* (on the given energy level), that is, the critical points of the integral form non-degenerate critical manifolds. The theory is based on the study of new topological invariants (rough molecules and marked molecules) of integrable systems discovered by Fomenko, Zieschang, Bolsinov and Matveev in [1]–[3]. The idea of the theory is to associate with every integrable system a certain invariant, which is, in fact, a labelled graph and has been called by us a t-molecule. Central to the theory is the assertion that two non-singular integrable systems are topologically orbitally equivalent if and only if the corresponding t-molecules are the same. Furthermore, it turns out that these invariants can be computed successfully in many concrete problems, leading to the results stated above.

Once continuous (that is, topological) orbital equivalence of the Euler and Jacobi systems is discovered, the natural question arises of whether or not the systems are topologically *adjoint*, that is, exactly equivalent. Perhaps there is a homeomorphism from one space to the other taking any exact solution into an exact solution (with the time preserved).

It turns out that this is not the case.

First of all, let us explain that the Euler and the Jacobi problems are both three-parameter problems. The geodesic flow on an ellipsoid is defined by the three squared half-axes a, b, c and the Euler system by the principal moments of inertia $1/A, 1/B, 1/C$ of the rigid body. If $a = b = c$ for an ellipsoid, then a sphere is obtained. But if $1/A = 1/B = 1/C$ for a rigid body, then we are dealing with a rigid *ball*. It is easily seen that the two systems are just the same in this case.

Theorem 16 (O. E. Orel). *The geodesic flow of any three-axial (that is, other than a sphere) ellipsoid restricted to any three-dimensional manifold of constant energy is not topologically adjoint to any Euler dynamical system for a rigid body.*

The idea of the proof of Orel's theorem is to compute and compare certain invariants of the dynamical systems under consideration.

As was demonstrated by Bolsinov and Fomenko in [15], [28], and [29], in the case at hand the topological orbital type of the integrable system is completely determined by two invariants k and l for the Jacobi problem and, respectively, K and L for the Euler problem. The point is that each of the dynamical systems to be compared has two stable periodic trajectories. The invariants k and l are the limits of the winding numbers of the dynamical systems as the Liouville tori approach these trajectories (in the limit the torus degenerates, turning into a circle). In the Jacobi problem these periodic trajectories have a clear geometrical meaning. They correspond to two stable closed geodesics, the equatorial plane sections of the ellipsoid in the directions perpendicular to its longest and shortest axes. In the case of the Euler problem the analogous periodic trajectories correspond to rotations of a rigid body about its maximum and minimum axes of inertia. Here the corresponding limits of winding numbers give the invariants K and L.

The invariants k, l and K, L are functions of a, b, c and A, B, C, respectively. Furthermore, the aforesaid relationships betwen the parameters of an 'orbitally equivalent' ellipsoid and a rigid body mean precisely that

$$k(a,b,c) = K(A,B,C) \quad \text{and} \quad l(a,b,c) = L(A,B,C).$$

But since we are now interested in the problem of comparing these two systems from the viewpoint of their adjointness, the two aforesaid invariants must be supplemented by at least three new ones, namely, the periods of the three closed singular trajectories. Two of them have been described above. These must be supplemented by one more *unstable* periodic trajectory, namely, the hyperbolic geodesic of an ellipsoid and, respectively, the unstable rotation of a rigid body about the middle axis of inertia. We denote these additional invariants by t_1, t_2, t_3 for the Jacobi problem and by T_1, T_2, T_3 for the Euler problem. As a result, the topological adjointness class of the Jacobi system is defined by a system of invariants certainly containing the five numbers

$$k,\ l,\ t_1,\ t_2,\ t_3,$$

and, respectively, for the Euler system

$$K,\ L,\ T_1,\ T_2,\ T_3.$$

It proves useful to state the explicit expressions for the periods of closed trajectories in the Euler and Jacobi problems.

In the Jacobi problem the period of a closed geodesic is simply equal to its length. For the periods in the Jacobi problem we thus obtain

$$\begin{aligned} t_1 &= \int_0^{2\pi} \sqrt{a\cos^2 t + b\sin^2 t}\, dt, \\ t_2 &= \int_0^{2\pi} \sqrt{a\cos^2 t + c\sin^2 t}\, dt, \\ t_3 &= \int_0^{2\pi} \sqrt{b\cos^2 t + c\sin^2 t}\, dt. \end{aligned}$$

In the case of the Euler problem the periods of motion along the three closed trajectories have the form

$$T_1 = \pi\sqrt{\frac{2}{C}}, \qquad T_2 = \pi\sqrt{\frac{2}{B}}, \qquad T_3 = \pi\sqrt{\frac{2}{A}}.$$

In both cases we assume that the energy H is equal to 1, that is, the energy level is fixed for both problems.

Since the Jacobi problem is a three-parameter one, assigning to every three-axial ellipsoid the aforesaid *five* numbers we obtain a smooth transformation of the three-dimensional set of all three-axial ellipsoids into the *five-dimensional* Euclidean space. As a reuslt, we obtain a 3-surface in R^5. We dentoe it by J^3. Following the same scheme for the Euler case, we also obtain a 3-surface E^3 in the same five-dimensional space R^5.

To prove that the Jacobi and Euler problems are not topologically adjoint, it suffices to verify that these two three-dimensional surfaces *do not intersect* one another in R^5, except at one point corresponding to the case $a = b = c$ and $A = B = C$, that is, the case of the standard 2-sphere.

After some analytic transformations this problem can be reduced to the following question: do two certain smooth curves intersect one another on the two-dimensional plane? They both start from the same point corresponding to the case $a = b = c$ (in the Jacobi problem) and $A = B = C$ (in the Euler problem). Theorem 16 follows from the analytically verifiable fact that these two curves *do not intersect* one another (except at one point, from which they start).

Finally, one more natural question arises: *are the Jacobi problem and the Euler case orbitally equivalent?*

This problem was completely solved by Bolsinov. The answer is that the Euler and Jacobi problems are not *smoothly* orbitally equivalent.

This answer can be obtained as follows. One must use the theory of smooth orbital classification of integrable Hamiltonian systems with two degrees of freedom constructed by Bolsinov [33]. Namely, one must compute smooth invariants of the systems under consideration and then compare them.

It turns out that it suffices to consider only one smooth invariant. Each of the two (Euler and Jacobi) systems to be compared has one unstable hyperbolic closed trajectory (to within the change to the opposite direction of motion). In fact, there are two such trajectories if we take the *direction* of motion along the hyperbolic circle into account. This hyperbolic trajectory has a natural smooth orbital invariant, namely, the multiplier, being an eigenvalue of the differential of the Poincaré map. For our two systems (Euler and Jacobi) these multipliers can be computed explicitly.

Here it proves more convenient for us to specify not the multiplier λ itself, but the inverse of its logarithm. For the Euler case and the Jacobi problem we denote this invariant by $M(A, B, C)$ and $m(a, b, c)$, respectively. It has the form

$$M(A, B, C) = -\frac{1}{\pi}\frac{B}{\sqrt{(C-B)(B-A)}},$$

$$m(a, b, c) = -\frac{\sqrt{\frac{b}{(c-b)(b-a)}}}{\int_0^{+\infty} \Phi(u, b)\, du}.$$

Now we consider two two-dimensional surfaces in the three-dimensional space R^3, which can be interpreted as the space of values of the invariants:

$$\mathcal{E} = \left\{ \begin{array}{c} x = K(A,B,C) \\ y = L(A,B,C) \\ z = M(A,B,C) \end{array} \right\}, \qquad \mathcal{J} = \left\{ \begin{array}{c} x = k(a,b,c) \\ y = l(a,b,c) \\ z = m(a,b,c) \end{array} \right\}.$$

These surfaces are images of the maps from the parameter spaces to the space of invariants.

If the position of the surfaces relative to one another is known, then we can draw certain conclusions.

If the surfaces do not intersect at all, then certainly not a single pair (rigid body, ellipsoid) exists for which the corresponding dynamical systems are smoothly equivalent (even in the sense of C^1-smoothness, since M and m are, as is easily seen, C^1-invariants).

If the surfaces are equal to one another, then this becomes a strong argument supporting the possibility that the systems under consideration are smoothly equivalent, since such an equality can hardly be accidental. Of course, in this case the study would need to be continued: we would have to compare all the remaining smooth invariants.

If the surfaces intersect one another along a curve, it means that the systems under consideration are not smoothly equivalent, as a rule. But there are some 'exceptional' pairs (rigid body, ellipsoid) for which at least three invariants are the same. Then, in fact, we would have to compare all the remaining smooth invariants for these exceptional pairs. Besides, in this situation it would be natural to go over to studying the problem of C^1-equivalence. It can be demonstrated that, apart from the above three invariants, there is precisely one more C^1-invariant. To answer the question fully it would be sufficient to analyse the relative position of two two-dimensional surfaces in the four-dimensional space of invariants. Their points of intersection would correspond to pairs of C^1-equivalent systems.

This is a possible general scheme of analysis, which can also be applied in other situations. In our case, because the explicit formulae are quite complex, Bolsinov carried out a computer-aided study of the problem in collaboration with Professor Dullin (Institute of Theoretical Physics, Bremen University, Germany).

The surfaces $\mathcal{E}$ (the Euler case) and $\mathcal{J}$ (the Jacobi case) have been constructed with the aid of a computer. It turned out that the qualitative beahviour of these surfaces is very similar. They both are graphs of certain functions $z = z_{\mathcal{E}}(x,y)$ and $z = z_{\mathcal{J}}(x,y)$.

This numerical study indicates that the surfaces $\mathcal{E}$ (the Euler case) and $\mathcal{J}$ (the Jacobi case) do not intersect one another: the difference $z_{\mathcal{E}}(x,y) - z_{\mathcal{J}}(x,y)$ is always negative, asymptotically approaching zero as the surfaces go off to infinity.

This method is of course just a numerical experiment, on which one cannot base a formulation of the result as a rigorously proved theorem (the surfaces may intersect one another somewhere far away). However, it is certainly a rigorous result that these two surfaces are not the same. In particular, this means that continuously orbitally equivalent pairs (rigid body, ellipsoid) cannot, as a rule, be smoothly equivalent (even in the sense of C^1-equivalence).

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