Amer. Math. Soc. Transl.
(2) Vol. 168, 1995

Various Aspects of n-Dimensional Rigid Body Dynamics

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Before we start to consider the motion of a "multi-dimensional body," it is useful to answer a question usually ignored by mathematicians dealing with this subject. Namely, is there any sense in such an occupation? Concerning the physical aspect, the answer seems to be negative. It does not even matter that the space in which we live is definitely three-dimensional. The main reason is deeper. As early as at the end of the last century, Paul Erenfest, the prominent physicist of the time, noted that in the n-dimensional space rigid bodies and even matter itself, as we understand it, could not exist. Such an unexpected conclusion is based on the natural assumption of universality of the energy conservation law and its mathematical expression, the Gauss–Ostrogradskiĭ principle. It follows from the latter that in the case $n \neq 3$ any central force must vary with the distance from an attracting center according to a law which is different from the Newtonian inverse square law. By the known theorems of dynamics, this implies that stable stationary orbits in the classical and quantum two body problems—the simplest models of the atom—cannot exist.

However, one should not hurry with the conclusion that the n-dimensional top and similar objects are no more than toys for mathematicians. There are several examples when equations of multi-dimensional dynamics can describe real mechanical systems.

Besides, while various Lie group constructions allow one to "see" multi-dimensional systems, studying such systems gives a remarkable opportunity to understand the properties of abstract groups better. In the present article, we consider multi-dimensional systems as the main source of new interesting integrable problems.

§1. Momentum theorem

In classical mechanics the term "momentum" is usually associated with a linear momentum (impulse), or an angular momentum, or a momentum of a force (torque) relative to an axis or a point. At the same time, this term has a universal interpretation and can be related to various geometric objects playing a major role in the description of dynamical systems.

Let M^n be the configuration space of a mechanical system with n degrees of freedom. Suppose an arbitrary Lie group $\mathfrak{G}$ acts on M^n. Let g be the Lie algebra of $\mathfrak{G}$ and g^* the space of linear functions on g (the dual space). To each vector

1991 *Mathematics Subject Classification*. Primary 70E99.

$Y \in g$ there corresponds a one-parameter subgroup $\mathfrak{G}_Y^\alpha \subset \mathfrak{G}$, $\alpha \in \mathbb{R}$, whose action on M^n determines the tangent vector field

$$v_Y(x) = \frac{d}{d\alpha} \mathfrak{G}_Y^\alpha(x)\bigg|_{\alpha=0}. \tag{1.1}$$

The mapping $Y \to v_Y$ is a homomorphism of the Lie algebra g to the Lie algebra of all vector fields on M^n (if $Y \to v_Y$ and $Z \to v_Z$, then $v_{[Y, Z]} = [v_Y, v_Z]$).

Let $T: TM^n \to \mathbb{R}$ be the kinetic energy of the mechanical system, which defines the inertial properties of the system. Here TM^n is the tangent bundle of M^n. In the applications, T is usually a positive definite quadratic form. We may think of it as a Riemannian metric on M^n. Define the function

$$\mathcal{I}_{\mathfrak{G}}(x, \dot{x} \mid Y) = \left(\frac{\partial T}{\partial \dot{x}}, v_Y\right) = \sum_{i=1}^n \frac{\partial T}{\partial \dot{x}_i}(v_Y)_i. \tag{1.2}$$

Under changes of the local (generalized) coordinates $(x_1, \dots, x_n)$, the collection of the derivatives $\partial T/\partial \dot{x}_i$ transforms according to the covariant law. Therefore, the function $\mathcal{I}_{\mathfrak{G}}(x, \dot{x} \mid Y)$ does not depend on the choice of coordinates. Besides, it is a linear function in Y, so $\mathcal{I}_{\mathfrak{G}} \in g^*$.

The *kinetic momentum* of the mechanical system *relative to the Lie group* $\mathfrak{G}$ is, by definition, the mapping $\mathcal{I}_{\mathfrak{G}}: TM^n \to g^*$ that assigns to each point $(x, \dot{x})$ of the phase space TM^n the linear function $\mathcal{I}_{\mathfrak{G}}(Y)$ on the algebra g. Let $y_1, \dots, y_n$ be a basis in g and $f^1, \dots, f^n$ be the dual basis in g^* with respect to the bilinear form $\langle \cdot, \cdot \rangle$ on $g \times g^*$ such that $\langle y_i, f^j \rangle = \delta_i^j$. Then the function $\mathcal{I}_{\mathfrak{G}}(Y)$ is uniquely defined by the vector $K = K_1 f^1 + \dots + K_n f^n \in g^*$ such that

$$\mathcal{I}_{\mathfrak{G}}(Y) = \langle Y, K \rangle_g. \tag{1.3}$$

The vector K is also called the kinetic momentum[1].

EXAMPLE 1. Let the mechanical system be a set of mass points $m_1, \dots, m_N$ with the position vectors $r_1, \dots, r_N \in \mathbb{R}^3$. Consider the action of the Lie group $E(3)$ on the configuration space $M^n = \{r_1, \dots, r_N\}$, $n = 3N$. Recall that $E(3)$ is the group of all rigid motions of three-dimensional Euclidean space $\mathbb{R}^3$, i.e., compositions of rotations and translations of this space:

$$r_k \to R r_k + s, \qquad k = 1, \dots, N.$$

Here $R: \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal rotation matrix, $s \in \mathbb{R}^3$ is a translation vector.

The action of a general one-parameter subgroup $\mathfrak{R}(\alpha)$ of $E(3)$ is realized as a helical motion of the space $\mathbb{R}^3$, i.e., it is the rotation about a fixed axis l through the angle αw ($w = \text{const}$) followed by the translation $s = \alpha v_l$ along the same axis:

$$r_k \to R_l(\alpha)(r_k - d) + d + \alpha v_l, \qquad k = 1, \dots, N.$$

[1] In the sequel, instead of saying "the kinetic momentum relative to the group $SO(3)$ (or even the group $SO(n)$)", we shall use the classical term "angular momentum".

Here $l = \{e_l t + d \mid t \in \mathbb{R}\}$, where e_l is the unit vector along this axis, $d \in \mathbb{R}^3$ is a constant translation vector, v_l is a velocity vector, α is the parameter of the subgroup. Using homogeneous coordinates in $\mathbb{R}^3$, we can write

$$r_k \to \mathfrak{R}(\alpha) r_k ,$$

$$\mathfrak{R}_l(\alpha) = \begin{pmatrix} R_l(\alpha) & d - R_l(\alpha) d + \alpha v_l \\ 0 & 1 \end{pmatrix}, \qquad r_k = (r_{k1}, r_{k2}, r_{k3}, 1)^T.$$

The element of the Lie algebra $e(3)$ generating the given one-parameter subgroup is represented by the *screw velocity* matrix

$$\text{(1.4)} \qquad \mathfrak{W} = \frac{\partial}{\partial \alpha} \mathfrak{R} \mathfrak{R}^{-1} = \begin{pmatrix} \Omega & v_l - \Omega d \\ 0 & 0 \end{pmatrix}, \qquad \Omega = \frac{\partial}{\partial \alpha} R(\alpha) R^{-1}(\alpha),$$

where Ω is a skew-symmetric 3×3 matrix (an element of the Lie algebra $so(3)$): $\Omega_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\omega_\gamma$, $\alpha, \beta, \gamma = 1, 2, 3$, and $\omega = w e_l = (\omega_1, \omega_2, \omega_3)^T$ is the angular velocity vector directed along the axis l.

According to (1.1), on $M^n = \{r_1, \dots, r_N\}$ this matrix generates the vector field

$$v_{\mathfrak{W}} = \{\omega \times (r_1 - d) + v_l, \dots, \omega \times (r_N - d) + v_l\} = \{\mathfrak{W} r_1, \dots, \mathfrak{W} r_N\}.$$

Using the expression for the kinetic energy

$$T = \frac{1}{2} \sum_{k=1}^{N} m_k (\dot{r}_k, \dot{r}_k)$$

and the definition (1.2), we obtain the following linear function on $e(3)$:

(1.5)

$$\mathcal{I}_{E(3)}(r, \dot{r} \mid \mathfrak{W}) = \sum_{k=1}^{N} m_k (\dot{r}_k, \omega \times (r_k - d) + v_l) = (\omega, M_d) + (v_l, p),$$

$$M_d = \sum_{k=1}^{N} (r_k - d) \times m_k \dot{r}_k, \qquad p = \sum_{k=1}^{N} m_k \dot{r}_k .$$

We see that M_d is the angular momentum of the system relative to the origin displaced at the vector d, whereas p is the total momentum of this system.

Now for any two matrices

$$\mathfrak{W}_1 = \begin{pmatrix} \Omega_1 & V_1 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{W}_2 = \begin{pmatrix} \Omega_2 & V_2 \\ 0 & 0 \end{pmatrix}, \qquad \Omega_1, \Omega_2 \in so(3), \; V_1, V_2 \in \mathbb{R}^3,$$

we define the Euclidean bilinear form $\langle \cdot , \cdot \rangle$ on $e(3) \times e^*(3)$ as follows

$$\text{(1.6)} \qquad \langle \mathfrak{W}_1, \mathfrak{W}_2 \rangle = -\frac{1}{2} \operatorname{tr}(\Omega_1 \Omega_2) + (V_1, V_2).$$

The first term at the right-hand side is the Killing form of the Lie algebra $so(3)$, i.e., the scalar product invariant with respect to the adjoint action of the rotation group $SO(3)$. Comparing the expressions (1.6) and (1.5), one can rewrite the latter in the form

$$\mathcal{I}_{E(3)}(\mathfrak{W}) = \langle \mathfrak{W}, K \rangle,$$

$$K = \begin{pmatrix} \widehat{M_d} & p \\ 0 & 0 \end{pmatrix} \in e^*(3), \qquad (\widehat{M_d})_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}(M_d)_\gamma . \tag{1.7}$$

According to definition (1.3), the matrix K represents the kinetic momentum of the system relative to the group $E(3)$.

In the special case $s = 0$, the group $E(3)$ is reduced to the group $SO(3)$, the corresponding one-parameter subgroup $R(\alpha)$ is generated by the angular velocity matrix Ω, and from (1.5) we obtain

$$\mathcal{I}_{SO(3)}(\Omega) = (\omega, M_0) = -\frac{1}{2}\operatorname{tr}(\Omega\widehat{M_0}), \qquad (\widehat{M_0})_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}(M_0)_\gamma,$$

i.e., the kinetic momentum relative to the group $SO(3)$ is represented by the 3×3 matrix $\widehat{M_0} \in so^*(3)$, which is isomorphic to the vector M_0 of the classical angular momentum relative to the origin O.

The action of the group $\mathfrak{G}$ on M^n may depend on time t. An example is the group of rotations of Euclidean space about a moving axis. In this case the vector field (1.1) depends on t, which should be regarded as a parameter.

Note that the action of $\mathfrak{G}$ on M^n extends naturally to an action on TM^n. Therefore, one can speak about invariants $f: TM^n \to \mathbb{R}$ of the action of the group $\mathfrak{G}$: these are the functions that are constant on the orbits of the action of this group on TM^n. The condition for a function $f(x, \dot{x})$ to be invariant has the form

$$\frac{d}{d\alpha} f(\mathfrak{G}^\alpha_Y(x, \dot{x}))\Big|_{\alpha=0} = \left(\frac{\partial f}{\partial \dot{x}}, \dot{v}_Y\right) + \left(\frac{\partial f}{\partial x}, v_Y\right) \equiv 0 \quad \text{for all } Y \in g. \tag{1.8}$$

If $f = T$ is the kinetic energy of a mechanical system and $\partial T/\partial x = 0$, then, in view of (1.2), (1.3), condition (1.8) can be represented in the form

$$\langle \dot{Y}, K \rangle = 0 \quad \text{for all } Y \in g. \tag{1.9}$$

For example, let $\mathfrak{G}$ be the group S_l of translations of the space $\mathbb{R}^3$ along the moving axis l with unit vector $e(t)$. The group S_l can be regarded as a one-parameter subgroup of the translation group $\mathbb{R}^3$. Put $g = \mathbb{R}^3$ in (1.9). Then the invariance condition for the function $T(\dot{x})$ takes the simple form

$$(p, \dot{e}) = 0. \tag{1.10}$$

A less trivial example arises when $\mathfrak{G}$ is the group $R_l = SO(2)$ of rotations of $\mathbb{R}^3$ about the same moving axis that passes through a moving point with the position vector $s(t)$. Similarly, the group R_l can be considered as a one-parameter subgroup of the group $E(3)$ generated by the element $\mathfrak{W}_l$ of the algebra $e(3)$ (compare with (1.4)):

$$\mathfrak{W}_l = \begin{pmatrix} \Omega_l & -\Omega_l s \\ 0 & 0 \end{pmatrix}, \qquad \Omega_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\omega_\gamma . \tag{1.11}$$

Here $\omega_l = we(t)$ is the angular velocity of rotation about the axis l. According to (1.9), the invariance condition for the kinetic energy is $\langle \dot{\mathfrak{W}}_l , K \rangle_{e(3)} = 0$, where K is the kinetic momentum relative to the group $E(3)$ defined in (1.7). In the vector form, this condition transforms to

$$(p , (s \times e)^{\cdot}) + (M_O , \dot{e}) = 0 . \tag{1.12}$$

Here M_O is the angular momentum of the system relative to the origin O. Note that, as one may expect, the obtained condition depends only on the Plücker coordinates $(s \times e , e)$ of the axis l, but does not depend on the coordinates of the position vector s itself. In particular, if the axis l does not change its direction in space, then (1.12) takes the form

$$(e , \dot{s} \times \dot{r}_C) = 0 , \tag{1.13}$$

where $r_C = \Sigma m_k r_k / \Sigma m_k$ is the position vector of the mass center C. This condition was obtained by Chaplygin in connection with a generalization of the angular momentum theorem (the area theorem) in certain mechanical problems (1897, [7]). It is obviously fulfilled provided the axis l always passes through the mass center.

Now consider the general situation, when the following constraints are imposed on a mechanical system:

$$f_1(x , \dot{x}) = \cdots = f_m(x , \dot{x}) = 0 . \tag{1.14}$$

In applications, the functions f_i are usually linear in the velocities $\dot{x}$. Assume that the gradients $\partial f_1/\partial \dot{x} , \ldots , \partial f_m/\partial \dot{x}$ are linearly independent. If equations (1.14) can be reduced to the form

$$\dot{h}_1(x) = \cdots = \dot{h}_m(x) = 0 , \tag{1.15}$$

then the constraints and the mechanical system itself are called *holonomic* (following H. Herz).

A simple example of a nonholonomic system is a ball rolling without sliding on a horizontal plane (the velocity of the contact point equals zero): without violating the constraints, the ball can be transformed from any configuration to any other one.

The action of the group $\mathfrak{G}$ is said to be *consistent with the constraints* (1.14) if

$$\left(\frac{\partial f_i}{\partial \dot{x}} , v_Y \right) = 0 , \qquad \text{for all } Y \in g, i = 1 , \ldots , m. \tag{1.16}$$

In the case of integrable constraints (1.15), this is equivalent to the invariance of the functions $h_1 , \ldots , h_m$ with respect to the action of $\mathfrak{G}$ on M^n. The tangent vectors v_Y satisfying condition (1.14) are called *virtual velocities*.

The equations of motion of nonholonomic systems can be written in the form of Lagrange equations with *multipliers*

$$\left(\frac{\partial T}{\partial \dot{x}}\right)^{\cdot} - \frac{\partial T}{\partial x} = F + \sum_{i=1}^{m} \lambda_i \frac{\partial f_i}{\partial \dot{x}_i}, \tag{1.17}$$

where $F = (F_1, \dots, F_n)$ are *generalized forces*. These equations, together with the constraint equations (1.14), constitute a closed system for determining the generalized coordinates x and the multipliers λ_i as functions of time.

As well as the gradients $\partial T/\partial \dot{x}$, the generalized force F is a covector. Hence, by analogy with (1.2) and (1.3), it is natural to define the mapping

$$\Phi_{\mathfrak{G}}(x, \dot{x} \mid Y) = \sum_{i=1}^{n} F_i (v_Y)_i = \langle \mathfrak{F}, Y \rangle .$$

This is the *torque relative to the group* $\mathfrak{G}$. The same name is given to the vector $\mathfrak{F} \in g^*$ that uniquely defines the linear function $\Phi_{\mathfrak{G}}(x, \dot{x} \mid Y)$.

THEOREM 1. *Suppose that the kinetic energy of the mechanical system is an invariant of the group* $\mathfrak{G}$*, and the action of this group is consistent with the constraints. Then for any* $Y \in g$ *the following relation holds*

$$\frac{d}{dt} \mathcal{I}_{\mathfrak{G}}(x, \dot{x} \mid Y) = \Phi_{\mathfrak{G}}(x, \dot{x} \mid Y), \tag{1.18}$$

which, in view of (1.16), (1.18), *implies the equation in* g^*

$$\frac{d}{dt} K = \mathfrak{F} . \tag{1.19}$$

This theorem goes back to Lagrange and Jacobi, who noticed the connection between conservation of momentum and angular momentum, and the groups of translation and rotation respectively. If the forces are potential ($F = -\partial V/\partial x$) and the potential energy $V : M^n \to \mathbb{R}$ is invariant with respect to the action of the group $\mathfrak{G}$, then (1.18) yields the first integral $\mathcal{I}_{\mathfrak{G}}(x, \dot{x}) = \text{const}$. This is a classical result of E. Noether (1918).

PROOF OF THEOREM 1. According to (1.16) and (1.17), we have

$$\left(\frac{\partial T}{\partial \dot{x}}\right)^{\cdot} v_Y - \frac{\partial T}{\partial x} v_Y = F v_Y ,$$

or, in another form,

$$\left(\frac{\partial T}{\partial \dot{x}} v_Y\right)^{\cdot} - \left[\frac{\partial T}{\partial \dot{x}} \dot{v}_Y + \frac{\partial T}{\partial x} v_Y\right] = F v_Y .$$

The expression in the square brackets vanishes by the invariance property of the function T.

COROLLARY. *Suppose that condition* (1.10) *is fulfilled and the vectors* $v_e = \{e, \dots, e\}$ *are virtual velocities on the configuration space* $M^n = \{r_1, \dots, r_N\}$, *i.e., the constraints admit infinitesimal translations of the point mass system as a rigid body along the axis* l. *Then, by* (1.18),

$$(p, e)^{\cdot} = (F, e).$$

In a similar way, if condition (1.12) *is fulfilled and the constraints admit infinitesimal rotations of the system as a rigid body about the (moving) axis* l, *then the following angular momentum theorem holds*

$$\dot{M}_l = (\mathcal{M}_F, e),$$

$$M_l = \left(\sum_{k=1}^{N}(r_k - d) \times m_k \dot{r}_k, e\right), \qquad \mathcal{M}_F = \sum_{k=1}^{N}(r_k - d) \times F_k. \tag{1.20}$$

EXAMPLE 2. As a remarkable application of Theorem 1, we consider Chaplygin's problem on a dynamically nonsymmetric ball rolling without sliding on a horizontal plane H (1903, [7]). The mass center C of the ball is assumed to coincide with its geometric center. Here the configuration space is the group $E(3)$, i.e., the set of the matrices

$$R = \begin{pmatrix} R & r_C \\ 0 & 1 \end{pmatrix},$$

where $R \in SO(3)$ is the orthogonal rotation matrix of the ball, r_C is the position vector of the mass center C with the initial point at the origin O of a fixed frame (we assume that O belongs to the plane H). Let P be the point of contact of the ball with the plane H. Then the condition for rolling without sliding has the form

$$v_P \equiv \dot{r}_C - \omega \times \rho\gamma = 0, \tag{1.21}$$

where ω, ρ are the angular velocity and the radius of the ball, γ is the vertical unit vector.

Converted to scalar form, condition (1.21) gives three nonholonomic constraints (in addition to one holonomic constraint $(r_C, \gamma) = \rho$).

PROPOSITION 2. *The constraints* (1.21) *admit infinitesimal rotations of the ball around the contact point* P.

The proof is obvious: under any such rotations one has $v_P = 0$, which coincides with the condition (1.21).

However, it is useful to consider also the following formal proof relying directly on the invariance condition (1.28). Since the constraints (1.21) depend neither on the orientation of the ball nor on its position on the plane H, it is sufficient to verify this invariance condition for only one configuration (i.e., at a certain point on the group $E(3)$). Assume that this configuration (point) is the identity of the group. Picking the expressions for the screw velocity (1.4) and the bilinear form (1.6), then taking into account the relation $r_P = r_C - \rho\gamma$, one can represent the constraint equations (1.21) in the following way

$$\langle N_\alpha , \mathfrak{W} \rangle_{e(3)} = 0 , \qquad \alpha = 1, 2, 3,$$

$$\mathfrak{W} = \dot{R}R^{-1} = \begin{pmatrix} \Omega & \dot{r}_c - \Omega r_c \\ 0 & 0 \end{pmatrix} \in e(3), \qquad N_\alpha \in e^*(3),$$

$$N_1 = \begin{pmatrix} 0 & R_2 & R_3 & 1 \\ -R_2 & 0 & 0 & 0 \\ -R_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & -R_1 & 0 & 0 \\ R_1 & 0 & R_3 & 1 \\ 0 & -R_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$N_3 = \begin{pmatrix} 0 & 0 & -R_1 & 0 \\ 0 & 0 & -R_2 & 0 \\ R_1 & R_2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where R_1, R_2, R_3 are the projections of the position vector r_P to the fixed axes, and the form $\langle \cdot , \cdot \rangle_{e(3)}$ is specified in (1.6).

Using the terminology introduced above, one may call the matrices N_α the momenta of constraints (1.21) relative to the action of the group $E(3)$. These stand for the covectors $\partial f_i / \partial \dot{x}$ in (1.16). According to (1.11), the velocity vector $v_{\mathfrak{W}}$ in $e(3) = T_e E(3)$ induced by infinitesimal rotations around the point P is represented by matrices of the form

$$\mathfrak{W}_l = \begin{pmatrix} \Omega_l & -\Omega_l r_P \\ 0 & 0 \end{pmatrix} \in e(3)$$

(the axis l passes through P). It follows that

$$\langle N_\alpha , \mathfrak{W}_l \rangle_{e(3)} = 0 \qquad \text{for any } l ,$$

i.e., condition (1.16) is satisfied on $T_e E(3)$ and, as a consequence, on the whole tangent bundle $TE(3)$.

Besides, for any axes l passing through the contact point P, Chaplygin's condition (1.13) is also satisfied. Since the torque of the constraint reaction at P relative to P is zero, Theorem 1 and equations (1.20) imply that the angular momentum of the ball K_P relative to the same point turns out to be a fixed vector in space.

Then, according to a well-known theorem of mechanics, we may write

$$K_P = K_C + \rho\gamma \times p , \qquad p = mv_C , \tag{1.22}$$

where m is the mass of the ball, K_C is its angular momentum relative to the mass center C.

In the moving axes attached to the ball, the momentum K_P satisfies the equations

$$\dot{K}_P + \omega \times K_P = 0 , \tag{1.23}$$

where, in view of (1.21), (1.22), one can put

$$K_P = I\omega + D\gamma \times (\omega \times \gamma) , \qquad D = m\rho^2 ,$$

I being the inertia tensor of the ball relative to the C. Then equations (1.23), together with the Poisson equations, form a closed system for determining $\omega(t)$ and $\gamma(t)$:

$$\begin{aligned} \Lambda\dot{\omega} &= \Lambda\omega \times \omega + (D/\varphi)((\Lambda\omega \times \omega)\Lambda^{-1})\gamma\,, \\ \dot{\gamma} &= \gamma \times \omega\,, \qquad \varphi = 1 - D(\gamma\,, \Lambda^{-1}\gamma)\,. \end{aligned} \tag{1.24}$$

From (1.23) and (1.24) it follows immediately that the system has three independent geometric integrals

$$\begin{aligned} (K\,, K) &= (\Lambda\omega\,, \Lambda\omega) - 2D(\Lambda\omega\,, \gamma) + D^2(\omega\,, \gamma)^2\,, \\ (K\,, \gamma) &= (I\omega\,, \gamma)\,, \qquad (\gamma\,, \gamma) = 1\,, \end{aligned} \tag{1.25}$$

as well as the kinetic energy integral

$$(K\,, \omega) = (\Lambda\omega\,, \omega) - D^2(\omega\,, \gamma)^2. \tag{1.26}$$

Besides, as shown by Chaplygin [7], in the phase space $(\omega\,, \gamma)$ the system possesses an integral invariant with density

$$\mu = \sqrt{\varphi}\,. \tag{1.27}$$

Therefore, by the well-known Jacobi theorem on the last multiplier, equations (1.24) are integrable by quadratures.

After Chaplygin, various integrable mechanical generalizations of the problem were found in [8, 13, 17] (see also Example 3 for a multi-dimensional generalization).

§2. Multi-dimensional dynamics

Now we proceed to the generalized Euler problem concerning the free motion of an n-dimensional rigid body around the fixed point in $\mathbb{R}^n$. This problem can be regarded as a classical one, since the idea of such a generalization was stated by A. Cayley as early as in the middle of the 19th century [6]. At present there exists a great number of publications devoted to this problem. Nevertheless, using the momentum theorem, here we give a detailed derivation of the equations of motion in order to set the background for constructing more complicated multi-dimensional systems.

What can be taken for the configuration manifold of the n-dimensional rigid body? The answer to this question is not unique. One may think of this manifold as a set of all positions of the body. It is known to be the group $SO(n)$, i.e., the group of orthogonal rotation matrices $R(t)$ such that

$$r_t = R(t)\rho\,, \qquad r_t\,, \rho \in \mathbb{R}^n\,, \tag{2.1}$$

where ρ is the position vector of some fixed point of the body relative to some frame $\mathcal{B}$ attached to the body (i.e., $\rho = \text{const}$), while r_t represents the same vector relative to some fixed frame. The components of the matrix R stand for the redundant generalized coordinates on $SO(n)$. In this case there are no constraints imposed.

At the same time, the configuration manifold can be defined as a set $\mathcal{R}$ of the position vectors of all points of the body

$$\mathcal{R} = \{r_1, r_2, \dots, r_N\} = \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n.$$

It is obvious that this manifold is actually infinite-dimensional and the generalized coordinates $r_1, r_2, \dots, r_N$ are strongly redundant in view of a great number of constraints fixing distances between all points of the body, as well as their distances from a fixed point O. These constraints are obviously invariant with respect to the action of the group $SO(n)$.

In the sequel we shall operate with both configuration varieties.

Differentiating (2.1), we can write

$$\dot{r}_t = \Omega_s r_t, \qquad \Omega_s = \dot{R}R^{-1}.$$

As in the three-dimensional case, the matrix Ω_s is called the angular velocity of the body relative to the fixed frame (the spatial angular velocity). This matrix turns out to be skew symmetric and, thereby, it is a vector in the Lie algebra $so(n)$.

Now let v be the velocity vector of a fixed point taken in the moving frame $\mathcal{B}$: $v = R^{-1}\dot{r}_t$. Then

$$v = \Omega_c \rho, \qquad \Omega_c = R^{-1}\dot{R}, \tag{2.2}$$

where $\Omega_c \in so(n)$ is the angular velocity matrix in the frame attached to the body (the body angular velocity). It is said that Ω_c and Ω_s arise as a result of left and right displacements (multiplications by R^{-1}) of the tangent vector $\dot{R} \in T_R SO(3)$ to the algebra $so(n) = T_E SO(n)$ regarded as the tangent linear space at the identity E of the group. Conversely, on the group regarded as a manifold, any matrix Ω generates the left-invariant (right-invariant) tangent vector field $v(R) = R\Omega \in T_R SO(n)$ ($v(R) = \Omega R \in T_R SO(n)$), where R runs through the group $SO(n)$.

From (2.1) and (2.2) we find that the angular velocities in the space and in the body are related as follows

$$\Omega_s = R\Omega_c R^{-1}, \tag{2.3}$$

or, in the vector form,

$$\Omega_s = \mathrm{Ad}_R\, \Omega_c, \qquad \Omega_s, \Omega_c \in \mathbb{R}^m,$$
$$\mathrm{Ad}_R\colon \mathbb{R}^m \to \mathbb{R}^m, \qquad m = \dim(so(n)) = n(n-1)/2,$$

where Ad_R is the operator of the adjoint action of the group $SO(n)$ on the algebra $so(n)$.

Using (2.2), we write the kinetic energy of the n-dimensional body in the form

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{r}_i, \dot{r}_i) = \frac{1}{2}\sum_{i=1}^{N} m_i(\Omega_c\rho_i, \Omega_c\rho_i), \qquad \rho = \text{const}, \tag{2.4}$$

where $(\cdot\,,\,\cdot)$ is the Euclidean scalar product in $\mathbb{R}^n$.

Now consider the *right* action of the group $SO(n)$ on itself: $R \to RG$ (G runs through $SO(n)$). A one-parameter subgroup of this action is $R\exp(\Omega_c\alpha)$, $\alpha \in \mathbb{R}$, where Ω_c is an arbitrary fixed vector in the algebra. According to (1.1), on the group regarded as a configuration manifold, each subgroup generates the *left-invariant* vector field

$$\frac{d}{d\alpha} R\exp(\Omega_c\alpha)\Big|_{\alpha=0} = R\Omega_c\,.$$

On the configuration manifold $\mathcal{R}$ specified above, the same subgroup generates the vector field $\{R\Omega_c\rho_1\,,\,\dots\,,\,R\Omega_c\rho_N\}$. Then, by the general formula (1.3), the kinetic (angular) momentum of the body relative to the *right* action of the group $SO(n)$ is defined as follows

$$\mathcal{I}_{SO(n)} = \sum_{i=1}^{N}\left(\frac{\partial T}{\partial \dot r_i}\,,\, R\Omega_c\rho_i\right) = \sum_{i=1}^{N} m_i(v_i\,,\,\Omega_c\rho_i) = \langle\Omega_c\,,\,M_c\rangle\,,$$

$$M_c = \sum_i^N m_i(v_i r_i^T - r_i v_i^T) \in so^*(n)\,.$$

Here the bilinear form $\langle\cdot\,,\,\cdot\rangle$ on $so(n)\times so^*(n)$ is also the Killing form of the algebra $so(n)$:

$$\langle\Omega_1\,,\,\Omega_2\rangle = -\frac{1}{2}\operatorname{tr}(\mathrm{ad}_{\Omega_1}\,\mathrm{ad}_{\Omega_2}) \equiv -\frac{1}{2}\operatorname{tr}(\Omega_1\Omega_2) \equiv \sum_{i<j}^{n}(\Omega_1)_{ij}(\Omega_2)_{ij}$$

for all $\Omega_1\,,\,\Omega_2 \in so(n)$. Since $v_i^T = \rho_i^T\Omega_c^T = -\rho_i^T\Omega_c$, we obtain

$$M_c = I_c\Omega_c + \Omega_c I_c\,, \qquad I_c = \sum_i^N m_i\rho_i\rho_i^T = \text{const}\,, \tag{2.5}$$

or, in vector form, $M_c = \mathcal{A}_I\Omega_c\,,\ \mathcal{A}_I : \mathbb{R}^m \to \mathbb{R}^m$.

The skew-symmetric matrix M_c and the symmetric matrix I_c are called the *angular momentum of the body* and its *mass tensor* in the *moving frame* respectively. Then it is natural to call the symmetric operator $\mathcal{A}_I$ the *inertia tensor* of the n-dimensional rigid body. Similarly to I_c, it is a constant tensor. In certain axes of the moving frame appropriately chosen tensors I_c and $\mathcal{A}_I$ transform simultaneously to diagonal form:

$$I_c = \mathrm{diag}(I_1\,,\,\dots\,,\,I_n)\,, \qquad (M_c)_{ij} = (I_i + I_j)(\Omega_c)_{ij}\,.$$

In a similar way, the one-parameter subgroups $\exp(\Omega_s\alpha)R$, $\alpha \in \mathbb{R}$, $\Omega_s \in so(n)$ of the *left* action $R \to GR$ of $SO(n)$ generates on the manifold $\mathcal{R}$ the vector field $\Omega_s r_i$. To this field there corresponds the kinetic momentum

$$\mathcal{I}_{SO(n)} = \sum_i^N\left(\frac{\partial T}{\partial \dot r_i}\,,\, R\Omega_s r_i\right) = \sum_i^N m_i(\dot r_i\,,\,\Omega_s r_i) = \langle\Omega_s\,,\,M_s\rangle_{so(n)}\,,$$

$$M_s = I_s\Omega_s + \Omega_s I_s \in so^*(n)\,, \qquad M_s = \mathcal{A}_I\Omega_s\,.$$

Here M_s, I_s, and $\mathcal{A}_I$ represent the angular momentum, the mass tensor, and the inertia tensor of the body in the fixed frame (in the space).

Since the Killing form on $so(n)$ is invariant with respect to Ad-transformations, M_s and M_c are transformed just as the angular velocities:

$$(2.6) \qquad M_s = RM_cR^{-1} \quad \text{or} \quad M_s = \operatorname{Ad}_R M_c .$$

This enables us to regard Ω_c and M_c as vectors in the same space $so(n) = \mathbb{R}^m$. The operator $\mathcal{A}_I$ defines on $so(n)$ the nondegenerate scalar product $\langle\ ,\ \rangle_{\mathcal{A}}$:

$$\langle Z, Q\rangle_{\mathcal{A}} = \langle Z, \mathcal{A}_I Q\rangle = \langle \mathcal{A}_I Z, Q\rangle \qquad Z, Q \in so(n),$$

which, in turn, defines on the group the left-invariant metric $(\cdot\,,\cdot)_{\mathcal{A}}$

$$(v_1, v_2)_{\mathcal{A}} = \langle R^{-1}v_1, R^{-1}v_2\rangle_{\mathcal{A}} \qquad v_1, v_2 \in T_RSO(n).$$

Then the kinetic energy (2.4) is represented in the form

$$T = \frac{1}{2}(\dot{R}, \dot{R})_{\mathcal{A}},$$

and, therefore, it is a function on $TSO(n)$ invariant with respect to the left action of the group.

The condition of free motion of the body implies the absence of torque relative to the action of the group $SO(n)$. Then, by Theorem 1, the angular momentum M_s relative to *left* action of the same group is constant.

Differentiating (2.6) and taking into account (2.2) along with the condition $M_s = \text{const}$, we obtain the generalized Euler equations describing the evolution of the angular velocity and of the angular momentum *in the body*

$$(2.7) \qquad \dot{M}_c + [\Omega_c, M_c] = 0.$$

The substitution $M_c = I_c\Omega_c + \Omega_c I_c$ turns (2.7) into a closed system of m scalar equations

$$(2.8) \qquad (I_i + I_j)\dot{\Omega}_{ij} = (I_i - I_j)\sum_{k=1}^{n} \Omega_{ik}\Omega_{kj}, \qquad i < j = 1, \dots, n$$

(the subscript c is omitted).

These were first represented in the explicit form by Frahm (1874) [10], who considered them together with n^2 kinematic equations

$$(2.9) \qquad \dot{R} = R\Omega_c .$$

The latter can be used to determine the position of the body in $\mathbb{R}^n$, i.e., the coordinates in the group $SO(n)$. Frahm also found a collection of first integrals of the joint system (2.8), (2.9) in the form

$$(2.10) \qquad \sum_J |M_c|_J^J = \text{const}, \qquad k = 2, 4, \dots, 2[n/2],$$

$$(2.11) \qquad \sum_{\mu<s}^{n} (M_c)_{\mu s}\widehat{R}_{ij,\mu s} = \text{const}, \qquad i < j = 1, \dots, n,$$

where $J = \{i_1, \dots, i_k\}$, $i_1 < \dots < i_k$, is a multi-index of order k, $|M_c|_J^J$ is the corresponding k-order diagonal minor of M_c, and $\widehat{R}_{ij,\mu s}$ are the 2×2 minors of the rotation matrix R standing at the crossings of ith, jth columns and μth, sth rows. These integrals generalize the angular momentum integral and the area integral in the classical Euler problem.

Besides, it is easy to show that system (2.8) possesses the energy integral

$$\langle \Omega_c, M_c \rangle = l, \qquad l = \text{const}. \tag{2.12}$$

It is known that, as in the classical case $n = 3$, the Euler–Frahm equations (2.8), as well as the corresponding kinematic equations (2.9), are integrable, and the components Ω_{ij}, R_{ij} can be expressed in terms of theta-functions of complex time t. Moreover, as follows from [16], both systems remain integrable under the more general relation between the angular momentum and the angular velocity:

$$(M_c)_{ij} = \frac{a_i + a_j}{b_i + b_j} (\Omega_c)_{ij}, \tag{2.13}$$

where $a_1, \dots, a_n, b_1, \dots, b_n$ are arbitrary fixed parameters (when $b_i = I_i$, $a_i = I_i^2$, we come back to (2.5)). For the case $n = 4$, the integrability condition (2.13) was, in fact, obtained by Frahm, while the analytical study of the problem was performed by Schottky (1891) [21].

EXAMPLE 3. Using the construction introduced above, let us consider the generalized Chaplygin problem on an n-dimensional ball rolling without sliding on an $(n-1)$-dimensional hyperplane $\mathcal{H}$ in $\mathbb{R}^n$. As in the case $n = 3$, this is a nonholonomic system and its equation of motion cannot be obtained from any variational principle. The only way is to use the momentum theorem for the group $E(n)$, which is the configuration space for the ball. For the case of an unconstrained motion in $\mathbb{R}^n$, this theorem is the generalization of the classical theorem which asserts that the equations of motion of a rigid body split into the equations describing rotational motion of the body and the equations describing motion of its mass center as a mass point.

Let $\gamma \in \mathbb{R}^n$ be the unit vector normal to the hyperplane $\mathcal{H}$ and directed "upwards" (i.e., from $\mathcal{H}$ to the mass center C of the ball). Then, as before, ρ is the radius of the ball, Ω and $M = I\Omega + \Omega I \in so(n)$ are respectively its angular velocity and angular momentum matrices, $V_C \in \mathbb{R}^n$ is the velocity of the mass center which coincides with the geometric center of the ball, v_P is the velocity of the contact point P (here and below all the tensor objects are taken in the frame attached to the ball). Now replace the constraint $v_P = 0$ by the reaction $F \in \mathbb{R}^n$ acting at P. Then in the frame attached to the ball, we obtain the following equations

$$\dot{M} + [\Omega, M] = \mathcal{M}, \qquad m(\dot{v}_C + \Omega v_C) = F, \qquad \dot{\gamma} + \Omega\gamma = 0, \tag{2.14}$$

where $\mathcal{M} \in so^*(n)$, $\mathcal{M}_{ij} = \rho(F_i\gamma_j - F_j\gamma_i)$ is the torque of the reaction F relative to C.

The condition for rolling without sliding has the form

$$v_P = v_C - \Omega\gamma = 0. \tag{2.15}$$

Differentiating (2.15) and using the second equation in (2.14), we get

$$F = m\rho\dot{\Omega}\gamma\,.$$

Substituting this expression in (2.14), we obtain

$$\dot{M} + [\Omega\,,\, M] = D(\dot{\Omega}\Gamma + \Gamma\dot{\Omega})\,, \qquad D = m\rho^2\,,\ \Gamma_{ij} = \gamma_i\gamma_j\,,$$
$$\dot{\Gamma} + [\Omega\,,\,\Gamma] = 0\,.$$

It is easy to show that this system can be represented in the following compact commutative form

$$\text{(2.16)} \qquad \begin{aligned} &\dot{K} + [\Omega\,,\, K] = 0\,, \qquad K = I\Omega + \Omega I + D(\Gamma\Omega + \Omega\Gamma) \in so^*(n)\,,\\ &\dot{\Gamma} + [\Omega\,,\,\Gamma] = 0\,, \end{aligned}$$

where K and $I + D\Gamma$ are respectively the angular momentum of the ball and its mass tensor relative to the contact point P. So, we may introduce the inertia operator $\hat{\mathcal{A}}$: $so\,(n) \to so\,(n)$ as follows:

$$\text{(2.17)} \qquad K = \hat{\mathcal{A}}\Omega = (\mathcal{A}_I + \mathcal{A}_\Gamma)\Omega\,, \qquad \mathcal{A}_I\Omega = I\Omega + \Omega I\,,\ \mathcal{A}_\Gamma\Omega = \Gamma\Omega + \Omega\Gamma\,.$$

Equations (2.16) can be uniquely represented in terms of Ω, Γ. So they form a closed system for the determination of $\Omega(t)$, $\gamma(t)$.

It follows from (2.16) that $K_s = \mathrm{Ad}_R\,K$ is a constant vector in $so\,(n)$. Since γ is constant in the space, the system (2.16) has a set of geometric integrals

$$\text{(2.18)} \qquad \begin{aligned} &\operatorname{tr} K^s = \text{const}\,, \quad \operatorname{tr}(K^s\Gamma^l) = \text{const}\,, \quad \operatorname{tr}\Gamma = 1\,,\\ &s = 2\,,4\,,6\,,\,\dots\,,\ l = 1\,,2\,,3\,,\,\dots\,. \end{aligned}$$

These are generalizations of the classical integrals (1.25).

PROPOSITION 3. *The system* (2.16) *has the integral invariant*

$$\int \mu\, d\Omega\, d\gamma\,, \qquad \mu = \sqrt{\det \hat{\mathcal{A}}}\,.$$

In the classical case $n = 3$, the density μ coincides with the density (1.27) found by Chaplygin.

PROOF OF PROPOSITION 3. First note that the system (2.16) can be rewritten in the form

$$\text{(2.19)} \qquad \hat{\mathcal{A}}\dot{\Omega} = -\operatorname{ad}_\Omega \mathcal{A}_I\Omega\,,$$

$$\text{(2.20)} \qquad \dot{\Gamma} = -\operatorname{ad}_\Omega \Gamma\,.$$

Besides, in view of (2.17), (2.20),

$$\text{(2.21)} \qquad \dot{\hat{\mathcal{A}}} = [\mathcal{A}_\Gamma\,,\, \operatorname{ad}_\Omega]\,.$$

In order to calculate the divergence of the system in the phase space (Ω, γ)

$$\Delta = \sum_{i<j}^{n} \frac{\partial \dot{\Omega}_{ij}}{\partial \Omega_{ij}} + \sum_{i}^{n} \frac{\partial \dot{\gamma}_i}{\partial \gamma_i}, \tag{2.22}$$

we define the $m \times m$ matrix

$$U_{QS} = \frac{\partial (\widehat{\mathcal{A}}\dot{\Omega})_Q}{\partial \Omega_S},$$

where the indices Q, S range over the pairs (i, j), $1 \leqslant i < j \leqslant n$. From (2.19) we obtain

$$U = -\operatorname{ad}_\Omega \mathcal{A}_I + \operatorname{ad}_M, \qquad M = \mathcal{A}_I \Omega. \tag{2.23}$$

In view of the last equation in (2.14), the second sum in (2.22) vanishes. Therefore we may write

$$\Delta = \operatorname{tr}(\widehat{\mathcal{A}}^{-1}\{U\}), \qquad \{U\} = \frac{1}{2}(U + U^T).$$

(Since $\widehat{\mathcal{A}}$ is a symmetric matrix, the skew symmetric part of U does not contribute to Δ.) Then, in view of (2.21), (2.23), and the identity

$$\operatorname{tr}(\widehat{\mathcal{A}}^{-1}[\widehat{\mathcal{A}}, \operatorname{ad}_\Omega]) \equiv 0,$$

we obtain

$$\begin{aligned}\Delta &= \operatorname{tr}(\widehat{\mathcal{A}}^{-1}(U + U^T))/2 = \operatorname{tr}(\widehat{\mathcal{A}}^{-1}[\mathcal{A}_I, \operatorname{ad}_\Omega])/2 \\ &= -\operatorname{tr}(\widehat{\mathcal{A}}^{-1}[\mathcal{A}_\Gamma, \operatorname{ad}_\Omega])/2 = -\operatorname{tr}(\widehat{\mathcal{A}}^{-1}\dot{\widehat{\mathcal{A}}})/2.\end{aligned}$$

In conclusion, comparing this with the well-known identity

$$\frac{d}{dt}\det \widehat{\mathcal{A}} = \det \widehat{\mathcal{A}} \operatorname{tr}(\widehat{\mathcal{A}}^{-1}\dot{\widehat{\mathcal{A}}}),$$

we have

$$\Delta = -\frac{1}{2}\frac{d(\det \widehat{\mathcal{A}})/dt}{\det \widehat{\mathcal{A}}}.$$

Therefore, the function $\mu = \sqrt{\det \widehat{\mathcal{A}}}$ satisfies the equation for the density of the integral invariant:

$$\dot{\mu} + \Delta \mu = 0.$$

Finally, in addition to the geometric integrals (2.18), the equations of motion (2.16) possess the energy integral

$$H(\Omega) = \frac{1}{2}\operatorname{tr}(\Omega \widehat{\mathcal{A}} \Omega) \tag{2.24}$$

(compare with (1.26)). Indeed, taking into account (2.19), (2.21), and the property $\widehat{\mathcal{A}}^T = \widehat{\mathcal{A}}$, we obtain

$$\begin{aligned}\dot{H} &= \operatorname{tr}(\Omega\widehat{\mathcal{A}}\dot{\Omega} + \Omega\dot{\widehat{\mathcal{A}}}\Omega/2) = \operatorname{tr}(\Omega[\mathcal{A}_l\Omega\,,\,\Omega]) - \langle\Omega\,,\,(\mathcal{A}_\Gamma\operatorname{ad}_\Omega - \operatorname{ad}_\Omega\mathcal{A}_\Gamma)\Omega\rangle \\ &= \operatorname{tr}(\Omega[\mathcal{A}_l\Omega\,,\,\Omega]) + \operatorname{tr}(\Omega[\mathcal{A}_\Gamma\Omega\,,\,\Omega])\,,\end{aligned}$$

which equals zero, since $\operatorname{ad}_\Omega\Omega = 0$ and $\operatorname{tr}(\Omega[X\,,\,\Omega]) = 0$ for any matrix X.

It is natural to suppose that, similarly to the Manakov system, the equations of motion of the n-dimensional Chaplygin ball are integrable, and, apart from (2.18), (2.24), there exist other nontrivial integrals (integrals (2.18), (2.24) form a complete set only for the classical case $n = 3$). However, the proof of this conjecture is still unknown.

§3. Generalized Poinsot model

How can one obtain a qualitative picture of the motion of n-dimensional rigid body in the integrable Euler–Frahm case? It is obvious that even if they are known, the functions $R_{ij}(t)$ can hardly be useful in this situation.

Recall that in the case $n = 3$ the remarkable Poinsot model of motion exists. Namely, consider the inertia ellipsoid with fixed center O

$$\mathcal{V}\colon \{(x\,,\,Jx) = l\}\,, \qquad x \in \mathbb{R}^3\,,$$

which is attached to the body with inertia tensor J. According to this model, the ellipsoid $\mathcal{V}$ rolls without sliding on a fixed plane π. The latter is perpendicular to the angular momentum vector $M = J\omega$, which is also fixed in the space and is located at constant distance $l/|M|$ from the center O (Figure 1(a)). The point P of contact of the ellipsoid with the plane traces the curve on $\mathcal{V}$ called the *polodia*.

Indeed, due to the energy integral $(\omega\,,\,J\omega) = l$, the end of the angular velocity vector ω always lies on $\mathcal{V}$. Let π be the tangent plane of $\mathcal{V}$ at the contact point P. Then the normal vector

$$n = \frac{1}{2}\operatorname{grad}(x\,,\,Jx)\Big|_{x=\omega}$$

coincides with the angular momentum vector M. Besides, the distance between π and the fixed center O is also constant: $(\omega\,,\,n) = l$. It follows from the condition $v_P = \omega \times \overrightarrow{OP} = 0$ that the plane π remains fixed in space.

This model does not admit an immediate generalization to the n-dimensional case, since for even n the skew symmetric operator Ω is generally nondegenerate and, as a consequence, an n-dimensional rigid body does not necessarily have a rotation axis as the set of instantly fixed points. Therefore, a rolling-without-sliding model cannot be realized in this situation. The way out is the following: instead of rotation in $\mathbb{R}^n$, we can consider rotation in the m-dimensional space of the Lie algebra $so(n)$. To carry out this approach, we note that, in view of (2.3), (2.5), (2.6), the mass tensor and the inertia tensor in the space and in the body are related as follows

$$\begin{aligned} I_s &= RI_cR^{-1}\,, \qquad & I\colon \mathbb{R}^n \to \mathbb{R}^n\,, \\ \mathcal{A}_s &= \operatorname{Ad}_R\mathcal{A}_c\operatorname{Ad}_R^{-1}\,, \qquad & \mathcal{A}\colon \mathbb{R}^m \to \mathbb{R}^m\,. \end{aligned}$$

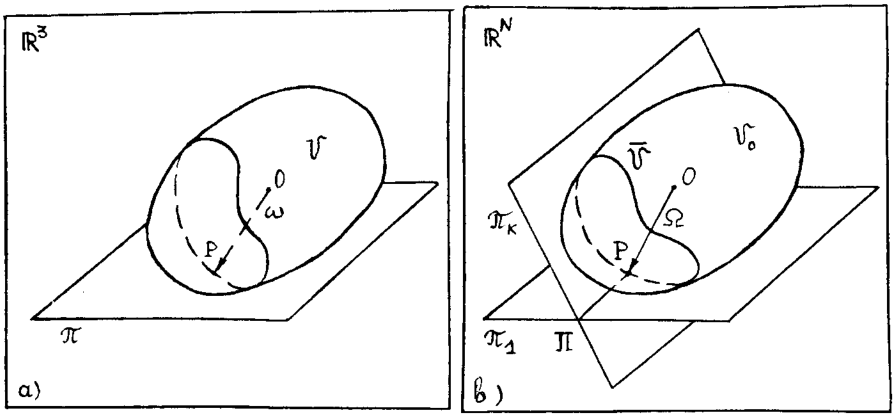

FIGURE 1

Comparing these two formulas, we see that it is natural to consider $\mathrm{Ad}_R \in SO(m)$ as the rotation matrix of an imaginary m-dimensional rigid body whose axes are attached to the eigen-axes of the tensor $\mathcal{A}_s$ in $\mathbb{R}^m$. Such an m-dimensional body is called the *kinematical body*.

Since the adjoint representation of the algebra $so(n)$ is exact, the position of the kinematical body in $\mathbb{R}^m$ uniquely defines the position of the "physical" n-dimensional body in $\mathbb{R}^n$.

Then from the equations

$$\dot{I}_s = [\Omega, I_s], \qquad \dot{\mathcal{A}}_s = [\mathrm{ad}_\Omega, \mathcal{A}_s]$$

we find that $\mathrm{ad}_\Omega \in so(m)$ plays the role of the angular velocity matrix of the kinematical body. In contrast to Ω, the operator ad_Ω is always degenerate and the "rotation axis" of the kinematical body is a whole r-dimensional linear space passing through the point O:

$$\mathrm{Ann}(\Omega) = \{d \in \mathbb{R}^m \mid \mathrm{ad}_\Omega d = 0\}, \qquad r = \mathrm{rank}(so(n)) = [n/2].$$

The space $\mathrm{Ann}(\Omega)$ is spanned by the matrices

$$\{\Omega, \Omega^3, \dots, \Omega^{2r-1}\}. \tag{3.1}$$

Using the Cayley–Hamilton theorem, one may show that the other odd powers Ω^i, $i > 2r - 1$, can be represented as linear combinations of the vectors (3.1). Furthermore, the Euler–Frahm equations (2.7) imply that for any degree i

$$\dot{M}_c^i + [\Omega, M_c^i] = 0,$$

and, similarly to M_s, the skew symmetric matrices $M_s^3 = \mathrm{Ad}_R M_c^3$, $M_s^5 = \mathrm{Ad}_R M_c^5, \dots$ turn out to be fixed vectors in the algebra $so(n)$ (among them only r vectors are independent). Therefore, these equations have r independent trivial momentum integrals

$$\langle M^{2i-1}, M^{2i-1}\rangle = \mathrm{const}, \qquad i = 1, \dots, r, \tag{3.2}$$

which are functions of r Frahm integrals (2.10).

Now consider the generalized Poinsot model illustrating the motion of the kinematical body. Let us deal with the simplest model corresponding to the "nonphysical" case $\Omega = UM + MU$, $U = \operatorname{diag}(a_1, \dots, a_n)$ (the general case can be handled in an analogous but more complicated way). Then equations (2.7), apart from the integrals (3.2), also possess the following nontrivial independent integrals

$$H_{mk}(M) = \operatorname{tr}\{M^m, U^k\} = h_{mk}, \qquad h_{mk} = \text{const}, \quad m = 2, 4, \dots, 2[(n-1)/2], \; k = 1, 2, \dots, n-m, \tag{3.3}$$

where $\{M^m, U^k\}$ are the homogeneous symmetric polynomials in the matrices M and U of degree m and k respectively (the integral H_{21}, up to the multiplication by a constant factor, coincides with the energy integral (2.12)). Now let X_{ij} be the projections of a vector $X \in \mathbb{R}^m = so(n)$ on the principal axes of $\mathcal{A}_s$. Let the inertia ellipsoid

$$\mathcal{V}_0 : \quad \left\{ \sum_{i<j}^{n} \frac{X_{ij}^2}{a_i + a_j} = h_{21} \right\}$$

be attached to the kinematical body as well as the surfaces $\mathcal{V}_{mk}$ $(k > 1)$, whose equations in the same principal axes arise from the integrals (3.3) as a result of the substitution

$$M_{ij} = \frac{X_{ij}}{a_i + a_j}.$$

The intersection of the surfaces $\mathcal{V}_0$, $\mathcal{V}_{mk}$ $(k > 1)$ define a surface $\widetilde{\mathcal{V}}$. Then we define the $(m - r)$-dimensional linear space $\Pi = \pi_1 \cap \dots \cap \pi_r$, where $\pi_i \subset \mathbb{R}^m$ is the hyperplane orthogonal to the fixed vector M_s^{2i-1} in the metric $\langle \cdot , \cdot \rangle$. The generalized Poinsot model is thus described by

THEOREM 4. *The set of all positions* $\{\mathrm{Ad}_R\}$ *corresponding to rotations* $\{R\}$ *of the n-dimensional rigid body coincides with the set of all positions in* $\mathbb{R}^m$ *of the surface* $\widetilde{\mathcal{V}}$ *with fixed center* $O = \{X = 0\}$, *which rolls without sliding on the fixed linear space* Π (*see the sketch in* Figure 1(b)). *The polodia, as a set on the surface* $\widetilde{\mathcal{V}}$ *traced by the point P, turns out to be not a single curve, but a whole d-dimensional torus, the joint level surface for the first integrals* (3.2), (3.3).

In the case $n = 3$ the surface $\widetilde{\mathcal{V}}$ coincides with the ellipsoid $\mathcal{V}_0$ and the linear space Π is merely the two-dimensional plane π_1, hence we come back to the classical Poinsot model.

COMMENTARY. The idea of the kinematical body was first discussed in [14]. It was proved that while this body rotates in the space $\mathbb{R}^m = so(n)$ according to the Euler–Frahm equations, the inertia ellipsoid $\mathcal{V}$ rolls without sliding on a fixed hyperplane $\pi \in so(n)$. However, in contrast to the three-dimensional case, for $n > 3$ the construction $\{\mathcal{V}, \pi\}$ fails to solve the "inverse problem", i.e., to recover the actual motion of the kinematical body. Namely, this construction admits superfluous positions of the inertia ellipsoid, and the contact point P on $\mathcal{V}_0$ covers the surface

$$\mathcal{V}_0 \cup \left\{ \sum_{i<j}^{n} \left(\frac{X_{ij}}{a_i + a_j} \right)^2 = \text{const} \right\},$$

which is more than the d-dimensional invariant torus. The construction $\{\widetilde{\mathcal{V}}, \Pi\}$ defined in Theorem 4 is free of this shortcoming. However, it also fails to reconstruct the motion of the kinematical body uniquely, since in the case $n > 3$ when we fix any instant rotation axis OP or even a k-dimensional linear space $\mathrm{Ann}(\Omega) \in \mathbb{R}^m$, we still leave some degrees of freedom for the surface $\widetilde{\mathcal{V}}$. Figuratively speaking, a certain "looseness" occurs. As a consequence, when dealing with the generalized Poinsot model based only on the first integrals, one cannot speak about the motion of the m-dimensional kinematical body but only about its positions.

PROOF OF THEOREM 4. In view of the definition of the surface $\widetilde{\mathcal{V}}$, the end of the vector $\Omega = \overrightarrow{OP}$ always belongs to this surface. Let the hyperplanes $\pi_1, \dots, \pi_r$ pass through P. The hyperplane π_1 is obviously tangent to $\mathcal{V}_0$ at P. Since $\Pi \subset \pi_1$, the linear space Π is also tangent to $\widetilde{\mathcal{V}} \subset \mathcal{V}_0$.

Let $n = M^{2i-1}/|M^{2i-1}|$, $|M^{2i-1}| = \sqrt{\langle M^{2i-1}, M^{2i-1}\rangle}$ be the unit normal vector of π. Then the latter is removed from the fixed center O at the distance $\langle \Omega, n_i\rangle = \langle UM + MU, M^{2i-1}\rangle/|M^{2i-1}|$, which, in view of the integrals (3.2), (3.3) for $k = 1$, is constant. Thus, by the condition $v_P = 0$, the linear space Π is fixed.

It follows that the end of the vector Ω in the principal axes of the ellipsoid $\mathcal{V}_0$ runs over the intersection of the surface $\widetilde{\mathcal{V}}$ with $\mathcal{V}_{m1}$, which is a d-dimensional invariant torus, the level surface for all first integrals (3.2), (3.3). The same holds for the trajectories of the Euler–Frahm equations in the phase space $\mathbb{R}^m = so(n)$.

§4. Euler–Poincaré equations with constraints

Let $v_1, \dots, v_n$ be linearly independent vector fields on M^n. Their commutators $[v_i, v_j]$ can be decomposed as follows

$$[v_i, v_j] = \sum c_{ij}^k(x) v_k, \qquad c_{ij}^k = -c_{ji}^k. \tag{4.1}$$

If $f(x)$ is a smooth function on M^n, then

$$\dot{f} = \left(\frac{\partial f}{\partial x}, \dot{x}\right) = \sum_{i=1}^n v_i(f)\omega_i,$$

where $v_i(f) = (\partial f/\partial x, v_i)$ is the derivative of f along v_i. The variables ω depend linearly on $\dot{x}$:

$$\dot{x}_k = \sum_{i=1}^n v_i(x_k)\omega_i, \qquad 1 \leqslant k \leqslant n. \tag{4.2}$$

The variables ω_i may not be derivatives of globally defined coordinate functions on M^n. For this reason the ω_i's are called *quasivelocities.*

The use of quasivelocities in mechanics is motivated by the fact that equations of motion written in terms of real velocities x often have a nonsymmetric and cumbersome form. As an example, we may recall the equations describing the rotation of a rigid body around a fixed point written in Euler's angles (θ, ψ, φ), which are globally defined coordinate functions on the group $SO(3)$.

Now rewrite the constraint equations (1.14) in the quasivelocities $\omega_1, \dots, \omega_n$:

$$f_1(\omega, x) = \cdots = f_m(\omega, x) = 0 \qquad (m < n). \tag{4.3}$$

Then the Lagrange equations (1.17) for the case of the motion in a potential force field with potential function $V(x)$ take the form

$$\left(\frac{\partial \mathcal{L}}{\partial \omega_i}\right)^{\cdot} = \sum_{j,k=1}^{n} c_{ji}^k \omega_j \frac{\partial \mathcal{L}}{\partial \omega_k} + v_i(\mathcal{L}) + \sum_1^m \lambda_s \frac{\partial f_s}{\partial \omega_i}, \tag{4.4}$$

where $\mathcal{L}$ is the Lagrangian $L(x, \dot{x}) = T(x, \dot{x}) - V(x)$ expressed in terms of ω, x. These equations (in the absence of constraints) were first obtained by H. Poincaré [20]. If the gradients

$$\frac{\partial f_1}{\partial \omega}, \dots, \frac{\partial f_m}{\partial \omega}$$

are linearly independent, then the multipliers λ_s can be represented as functions of x and ω.

In general, equations (4.4) do not represent a closed system, and they must be considered together with the geometric equations (4.2).

Now let M^n be a Lie group $\mathfrak{G}$, and $v_1, \dots, v_n$ be the independent left-invariant vector fields on $\mathfrak{G}$ generated, according to (1.1), by basis vectors $Y_1, \dots, Y_n$ in the Lie algebra g of the group $\mathfrak{G}$. Then the coefficients c_{ji}^k, called the structure constants of the algebra g, are fixed:

$$[Y_i, Y_j] = \sum_k c_{ji}^k Y_k,$$

which is equivalent to relations (4.1). In this case the velocity vector $\dot{x} = (\dot{x}_1, \dots, \dot{x}_n) \in T_x\mathfrak{G}$ is generated by left translations of the vector $\Omega = \omega_1 Y_1 + \cdots + \omega_n Y_n \in g$. Suppose that the Lagrangian $\mathcal{L}$ reduces to the kinetic energy ($L = T$) defined by a nondegenerate left-invariant metric $(\cdot, \cdot)_J$ on $\mathfrak{G}$. The latter is generated by the corresponding scalar product $\langle \cdot, \cdot \rangle_J$ on g. According to (4.2),

$$L = \frac{1}{2}(\dot{x}, \dot{x})_J = \frac{1}{2}\left(\sum_i v_i\omega_i, \sum_j v_j\omega_j\right)_J = \frac{1}{2}\sum_{i,j} J_{ij}\omega_i\omega_j.$$

Here $J_{ij} = (v_i, v_j)_J = \langle Y_i, Y_j \rangle_J = \text{const}$, because $v_1, \dots, v_n$ are left-invariant.

Consider the left action of the group $\mathfrak{G}$ on itself. Then one can consider the kinetic momentum relative to this action. According to the definition (1.2) and by (4.2), we obtain

$$\begin{aligned} K_s &= \mathcal{I}_{\mathfrak{G}}(x, \dot{x} \mid Y_s) = (\dot{x}, v_s)_J \\ &= \left\langle \sum_i \omega_i Y_i, Y_s \right\rangle_J = \sum_i \langle Y_i, Y_s \rangle_J \omega_i = \sum_i J_{si}\omega_i. \end{aligned}$$

Therefore, for an arbitrary vector $Y = y_1 Y_1 + \cdots + y_n Y_n \in g$, we have

$$\mathcal{I}_{\mathfrak{G}}(x, \dot{x} \mid Y) = \langle Y, K \rangle, \qquad K = \partial\mathcal{L}/\partial\omega. \tag{4.5}$$

Thus, according to (1.3), the vector $K = (K_1, \dots, K_n)^T \in g^*$ is precisely the kinetic momentum which was to be found.

It is interesting to study the case when, in addition to the Lagrangian $\mathcal{L}$, the constraint functions $f_1, \dots, f_m$ are also left-invariant, i.e., they do not depend explicitly on x. Under this condition the equations (4.4), (4.3) form a closed system on the Lie algebra g:

$$\dot{K}_i = \sum_{j,k=1}^{n} c_{ji}^k K_k \omega_j + \sum_{s=1}^{m} \lambda_s \frac{\partial f_s}{\partial \omega_i}, \qquad f_1(\omega) = \cdots = f_m(\omega) = 0. \tag{4.6}$$

For the case $\mathfrak{G} = SO(3)$, systems with left-invariant constraints were first studied by G. Suslov (1900, [22]). He considered rotations of a rigid body about a fixed point under the action of the following nonholonomic constraint: the projection of the angular velocity vector $\omega \in \mathbb{R}^3$ to a certain unit vector e_l *fixed in the body* always equals zero:

$$(\omega, e_l) = 0. \tag{4.7}$$

The left action of the group $SO(3)$ leaves the kinetic energy of the body and the constraint (4.7) invariant. Thus, one may call the equations (4.6) the *Euler–Poincaré–Suslov* equations (EPS).

EXAMPLE 4. Consider the EPS equations on the Lie algebra $so(n)$. How can one write a multi-dimensional analog of the condition (4.7)? In order to answer this question recall that, instead of rotations *about an axis* in the three-dimensional case, in an n-dimensional case we have to speak about infinitesimal rotations in the two-dimensional planes $e_i \wedge e_j$ spanned by the vectors of an orthonormal frame $e_1, \dots, e_n$. Suppose, without loss of generality, that in the condition (4.7) $e_l = (1, 0, 0)$. Then this condition can be redefined as follows: only infinitesimal rotations in the planes $e_1 \wedge e_2$, $e_1 \wedge e_3$ are allowed. Hence, it is natural to define a multi-dimensional analog of Suslov's condition in the following way: for an n-dimensional body only infinitesimal rotations in the planes $e_1 \wedge e_j$, $j = 1, \dots, n$, i.e., in the planes containing the vector e_1, are allowed. Therefore, we have the following constraints imposed on the components of the angular velocity matrix *in the body*

$$\Omega_{ij} = 0, \qquad i, j \geqslant 2, \tag{4.8}$$

or

$$\Omega = \begin{pmatrix} 0 & \Omega_{12} & \dots & \Omega_{1n} \\ -\Omega_{12} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\Omega_{1n} & 0 & \dots & 0 \end{pmatrix}. \tag{4.9}$$

According to (2.6), the angular momentum M is represented by the matrix $M = I\Omega + \Omega I$. By an appropriate orthogonal transformation which leaves the vector e_1

invariant, the symmetric mass tensor I can be reduced to the form

$$I = \begin{pmatrix} I_{11} & I_{12} & \dots & I_{1n} \\ I_{11} & I_{12} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I_{1n} & 0 & \dots & I_{nn} \end{pmatrix} \tag{4.10}$$

(the constraint equations (4.8) do not change under such a transformation). Therefore, equations (4.6) can be represented as follows

$$\dot{M} = [M, \Omega] + \Lambda, \qquad \Lambda = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \lambda_{23} & \dots & \lambda_{2n} \\ 0 & -\lambda_{23} & 0 & \dots & \lambda_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\lambda_{2n} & -\lambda_{3n} & \dots & 0 \end{pmatrix}, \tag{4.11}$$

where λ_{ij} are Lagrange multipliers. Taking into account (4.8) and (4.10), we obtain from (4.11) the following closed system for the entries $\Omega_{12}, \dots, \Omega_{1n}$

$$\begin{aligned} (I_{11} + I_{22})\dot{\Omega}_{12} &= I_{12}(\Omega_{13}^2 + \Omega_{14}^2 + \dots + \Omega_{1n}^2) \\ &\quad - (I_{13}\Omega_{13} + I_{14}\Omega_{14} + \dots + I_{1n}\Omega_{1n})\Omega_{12}, \\ (I_{11} + I_{33})\dot{\Omega}_{13} &= I_{13}(\Omega_{12}^2 + \Omega_{14}^2 + \dots + \Omega_{1n}^2) \\ &\quad - (I_{13}\Omega_{13} + I_{14}\Omega_{14} + \dots + I_{1n}\Omega_{1n})\Omega_{13}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (I_{11} + I_{nn})\dot{\Omega}_{1n} &= I_{1n}(\Omega_{12}^2 + \Omega_{13}^2 + \dots + \Omega_{1,n-1}^2) \\ &\quad - (I_{12}\Omega_{12} + I_{13}\Omega_{13} + \dots + I_{1,n-1}\Omega_{1,n-1})\Omega_{1n}. \end{aligned} \tag{4.12}$$

The equations for other Ω_{ij}'s can be used to determine the multipliers λ_{ij}, $i, j \geqslant 2$.

In the special case $I_{12} = \dots = I_{1n} = 0$, it follows from the equations (4.12) that $\Omega_{12}, \dots, \Omega_{1n} = \text{const}$ and the linear space in $so(n)$ spanned by the admissible velocities (4.9) is the eigenspace for the inertia operator $\mathcal{A}_I : so(n) \to so(n)^*$. In the general case, equations (4.12) have the energy integral

$$2H(\Omega) = (I_{11}+I_{22})\Omega_{12}^2 + (I_{11}+I_{33})\Omega_{13}^2 + \dots + (I_{11}+I_{nn})\Omega_{1n}^2 = h, \qquad h = \text{const}.$$

Thus, if $h > 0$, we obtain a dynamical system on an $(n-2)$-dimensional ellipsoid. Now put

$$F = (I_{11} + I_{22})I_{12}\Omega_{12} + (I_{11} + I_{33})I_{13}\Omega_{13} + \dots + (I_{11} + I_{nn})I_{1n}\Omega_{1n}.$$

Then the derivative of $F(\Omega)$ along the trajectories of the system (4.12) has the form

$$\dot{F} = \sum_{i<j}^{n} (I_{1i}\Omega_{1j} - I_{1j}\Omega_{1i})^2,$$

and it is positive everywhere in $so(n)$ except at the points of the line

$$\{\Omega_{12} = I_{12}\mu, \ldots, \Omega_{1n} = I_{1n}\mu, \ \mu \in \mathbb{R}\}. \tag{4.13}$$

These are the equilibrium points for the system (4.12). The line (4.13) meets the ellipsoid $H(\Omega) = h$ in the two diametrically opposed points S^-, S^+, in which the hyperplane $F(\Omega) = \text{const}$ is tangent to the ellipsoid. These points correspond to stable and unstable permanent rotations of the body in a certain 2-plane $v_1 \wedge v_2$ which is fixed in the space, as well as in the body (so called *plane rotations*).

Since $\dot{F} \geqslant 0$, all trajectories of the system (4.12) lying on one and the same ellipsoid are double-asymptotic: they tend to the points S^- and S^+ as $t \to \pm\infty$ (Figure 2(a)).

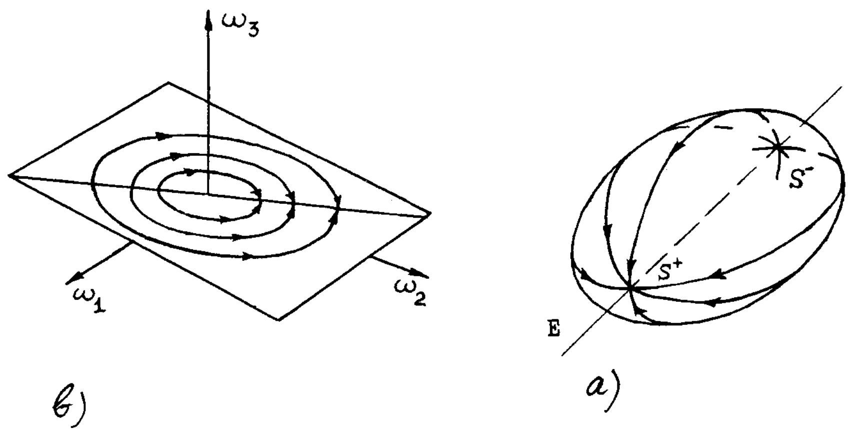

FIGURE 2

This picture represents an immediate generalization of the well-known phase portrait of the classical Suslov problem where the ellipsoid $\{H(\Omega) = h\}$ is reduced to an ellipse. The latter lies in the plane given by the equation (4.7) (Figure 2b). The motion of the n-dimensional rigid body is the asymptotic evolution from the permanent rotation in some 2-plane *fixed in the body* to the permanent rotation in the same 2-plane and with the same angular velocity, but in the opposite direction. We emphasize, however, that *in space* these rotations for $t \to -\infty$ and $t \to +\infty$ occur in different 2-planes.

As in the three-dimensional case, equations describing the position of the body in the space (i.e., the equations on the group $SO(n)$) seem to be nonintegrable.

What can one say about the analytic properties of the solutions of the system (4.12)? In the case $n = 3$, as it was noticed by Suslov, these solutions turn out to be meromorphic functions of the time t. More exactly, they are expressed in terms of fractional-rational functions of the form $\exp(bt)$ ($b = \text{const}$). (We leave the proof of this fact to the reader as an exercise.) In the case $n = 4$, the situation is different: the solutions of the system (4.12) generally branch in the complex plane. This follows from

PROPOSITION 5. *Let $n \geqslant 4$, $I_{12} \neq 0$, $I_{13} = \cdots = I_{1n} = 0$. Then the general solution of the system* (4.12) *is single-valued if and only if $I_{33} = \cdots = I_{nn} = 0$.*

PROOF OF THE NECESSITY. Let us use the asymptotic Kowalewski–Lyapunov method (see, for example, [24]). Note that if the conditions of Proposition 5 are fulfilled, the system (4.12) has the following particular solutions

$$\Omega_{12} = \alpha/t\,, \quad \Omega_{13} = \alpha/t\,, \quad \Omega_{14} = \cdots = \Omega_{1n} = 0\,, \tag{4.14}$$
$$\alpha = (I_{11} + I_{33})/I_{12}\,, \quad \beta^2 = -(I_{11} + I_{22})(I_{11} + I_{33})/I_{12}^2\,.$$

The corresponding Kowalewski exponents are

$$-1\,,\ 2\,,\ 1 - \frac{I_{11} + I_{33}}{I_{11} + I_{44}}\,,\ \ldots\,,\ 1 - \frac{I_{11} + I_{33}}{I_{11} + I_{nn}}\,.$$

The existence of the exponent 2 follows from the existence of the quadratic energy integral. According to the Lyapunov theorem, if the solutions are single-valued, then all the ratios $(I_{11} + I_{33})/(I_{11} + I_{ss})$, $s \geqslant 4$, must be integers. Apart from (4.14), equations (4.12) have other meromorphic solutions in which all the Ω_{1s}'s, $s \geqslant 3$, except one, are zero. In this case, performing elementary computations and using the Lyapunov theorem again, we came to the conclusion that the numbers $(I_{11} + I_{ss})/(I_{11} + I_{kk})$ for $k\,,\ s \geqslant 3$ must be integers as well. Then, since $I_{ss} \geqslant 0$, the equalities $I_{33} = \cdots = I_{nn}$ must hold.

The proof of the sufficiency is left to the reader.

Thus, in the general case when $I_{33}\,, \ldots\,, I_{nn}$ are all different, the solutions of (4.12) cannot be single-valued functions of complex time.

EXAMPLE 5. Now consider the more general case when the left-invariant constraints are defined as follows

$$\Omega_{ij} = 0\,, \qquad i\,, j > 2\,.$$

Suppose that the mass tensor I is diagonal, $I = \operatorname{diag}(I_1\,, \ldots\,, I_n)$. Then, in view of (4.6), the EPS equations take on the following simple form

$$\begin{aligned}
(I_1 + I_2)\dot{\Omega}_{12} &= (I_2 - I_1)(\Omega_{13}\Omega_{23} + \cdots + \Omega_{1n}\Omega_{2n})\,,\\
(I_1 + I_3)\dot{\Omega}_{13} &= (I_1 - I_3)\Omega_{12}\Omega_{23}\,, \ldots\,, (I_1 + I_n)\dot{\Omega}_{1n}\\
&= (I_1 - I_n)\Omega_{12}\Omega_{2n}\,,\\
(I_2 + I_3)\dot{\Omega}_{23} &= (I_3 - I_2)\Omega_{12}\Omega_{23}\,, \ldots\,, (I_2 + I_n)\dot{\Omega}_{2n}\\
&= (I_n - I_2)\Omega_{12}\Omega_{1n}\,.
\end{aligned} \tag{4.15}$$

These equations have $n - 1$ quadratic integrals

$$2H = \sum_{i \leqslant 2\,,\, j}^{n} (I_i + I_j)\Omega_{ij}^2\,,$$

$$F_1 = (I_1 + I_3)(I_2 - I_3)\Omega_{13}^2 + (I_2 + I_3)(I_1 - I_3)\Omega_{23}^2\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F_{n-2} = (I_1 + I_n)(I_2 - I_n)\Omega_{1n}^2 + (I_2 + I_n)(I_1 - I_n)\Omega_{2n}^2\,,$$

which enable one to solve the system (4.15) by quadratures.

Suppose, without loss of generality, that $I_1 > I_2 > \cdots > I_n$. Then the functions $F_1(\Omega), \dots, F_{n-2}(\Omega)$ are positive definite and we come to

PROPOSITION 6. *If* $c_1, \dots, c_{n-2} > 0$ *and*

$$c_1/(I_2+I_3) + \cdots + c_{n-2}/(I_2+I_n) < h, \tag{4.16}$$

then the integral surface

$$\mathcal{T}: \quad \{\Omega_{ij} \mid 2H(\Omega) = h,\ F_1(\Omega) = c_1, \dots, F_{n-2}(\Omega) = c_{n-2}\}, \qquad c_1, \dots, c_{n-2} = \text{const},$$

is the disjoint union of two $(n-2)$*-dimensional tori.*

PROOF. When $c_s > 0$, the curve $\{F_s = c_s\}$ on the plane $(\Omega_{1,s+2}, \Omega_{2,s+2})$ is an ellipse. Then, if condition (4.16) is fulfilled, Ω_{12} does not vanish on $\mathcal{T}$ and it is expressed in terms of the other Ω_{ij}'s up to sign flip. Therefore, a connected component of $\mathcal{T}$ is constructed as the topological product of circles.

Now define on $\mathcal{T}$ the angular variables $\varphi_1, \dots, \varphi_{n-2}$ by putting

$$\begin{aligned}\Omega_{1,s+2} &= \sqrt{\frac{c_s}{(I_1+I_{s+2})(I_2-I_{s+2})}}\sin(\varphi_s),\\ \Omega_{2,s+2} &= \sqrt{\frac{c_s}{(I_2+I_{s+2})(I_1-I_{s+2})}}\cos(\varphi_s),\end{aligned} \qquad s = 1, \dots, n-2, \tag{4.17}$$

and introduce the new time τ by the formula $d\tau = \Omega_{12}\,dt$. In view of the conditions of Proposition 1, $\Omega_{12} \neq 0$ on $\mathcal{T}$. Therefore, τ is a monotonic function of t, and, using (4.15), we obtain

$$\frac{d\varphi_1}{d\tau} = \nu_1, \ \dots, \ \frac{d\varphi_{n-2}}{d\tau} = \nu_{n-2}, \tag{4.18}$$

$$\nu_1 = \sqrt{\frac{(I_1-I_3)(I_2-I_3)}{(I_1+I_3)(I_2+I_3)}}, \ \dots, \ \nu_{n-2} = \sqrt{\frac{(I_1-I_n)(I_2-I_n)}{(I_1+I_n)(I_2+I_n)}}.$$

Equations (4.18) define a quasiperiodic motion on the $(n-2)$-dimensional tori with fixed frequencies $\nu_1, \dots, \nu_{n-2}$.

Going back to the original time t, we obtain the following system

$$\frac{d\varphi_1}{dt} = \frac{\nu_1}{F}, \ \dots, \ \frac{d\varphi_{n-2}}{dt} = \frac{\nu_{n-2}}{F}, \tag{4.19}$$

$$F^{-2} = \frac{1}{I_1+I_2}\Omega_{12}\left(h - \sum_{s=1}^{n-2}\left(\frac{c_s}{I_2-I_{s+2}}\cos^2(\varphi_s) - \frac{c_s}{I_1-I_{s+2}}\sin^2(\varphi_s)\right)\right).$$

This system has the integral invariant

$$\operatorname{mes}(\mathcal{D}) = \int_{\mathcal{D}} F\, d\varphi_1 \dots d\varphi_{n-2}.$$

As shown by Kolmogorov, for almost all frequencies $\nu_1, \dots, \nu_{n-2}$ equations (4.19) can be reduced to the form (4.18).

In conclusion, let us discuss the analytic properties of the solutions of system (4.15). Put $c_1, \dots, c_{n-2} = 0$. Then $\Omega_{14} = \Omega_{24} = \dots = \Omega_{1n} = \Omega_{2n} = 0$, and equations (4.15) reduce to a closed system for $\Omega_{12}, \Omega_{13}, \Omega_{23}$, which coincides with the equations of the integrable Euler problem. The solutions of the latter equations are known to be elliptic, i.e., single-valued functions of time t. So one may ask whether or not this property is valid for *all* solutions of (4.15)? It turns out that in general the answer is negative.

PROPOSITION 7. *All the solutions of equations* (4.15) *are single-valued functions on complex plane iff* $\nu_1 = \dots = \nu_{n-2}$.

PROOF. As above, we use the asymptotic Kowalewski–Lyapunov method. On the one hand, system (4.15) has the following particular solution

$$\Omega_{12} = \frac{\alpha_{12}}{t}, \quad \Omega_{13} = \frac{\alpha_{13}}{t}, \quad \Omega_{23} = \frac{\alpha_{23}}{t},$$
$$\Omega_{14} = \Omega_{24} = \dots = \Omega_{1n} = \Omega_{2n} = 0,$$
$$\alpha_{12} = \sqrt{\frac{(I_1+I_3)(I_2+I_3)}{(I_1-I_3)(I_3-I_2)}}, \quad \alpha_{13} = \sqrt{\frac{(I_1+I_2)(I_2+I_3)}{(I_2-I_1)(I_3-I_1)}},$$
$$\alpha_{13} = \sqrt{\frac{(I_1+I_3)(I_2+I_1)}{(I_1-I_3)(I_2-I_1)}}.$$

The Kowalewski exponents for this solution are

$$\rho_1 = -1, \quad \rho_2 = \rho_3 = 2, \quad \rho_{4,5} = 1 \pm \nu_2/\nu_1, \quad \dots, \quad \rho_{2n-4,2n-3} = 1 \pm \nu_{n-2}/\nu_1.$$

If general solution of (4.15) is single-valued, then the ratios $\nu_2/\nu_1, \dots, \nu_{n-2}/\nu_1$ must be integers. On the other hand, for each $3 \leqslant i \leqslant n$ these equations have also the particular solution

$$\Omega_{12} = \frac{\beta_{12}}{t}, \quad \Omega_{1i} = \frac{\beta_{1i}}{t}, \quad \Omega_{2i} = \frac{\beta_{2i}}{t}, \qquad \Omega_{1k} = \Omega_{2k} = 0 \quad (k \neq i)$$

(the definition of the β's is analogous to that of the α's). Then the condition for the corresponding Kowalewski exponents to be integers implies that the ratios $\nu_1/\nu_i, \nu_2/\nu_i, \dots, \nu_{n-2}/\nu_i$ must be integers as well. Since $\nu_s > 0$ for all s, we come to the condition $\nu_1 = \dots = \nu_{n-2}$.

Now let us prove the sufficiency of this condition. It turns out that if $\nu_1 = \dots = \nu_{n-2} = \nu$, then the solutions of (4.15) are elliptic functions of t. To show this, we use the new time τ. Then, from (4.18) we have $\varphi_s = \nu\tau + \varphi_s^0$, $\varphi_s^0 = \text{const}$ $(1 \leqslant s \leqslant n-2)$. Since $d\tau = \Omega_{12}\,dt$, in view of (4.19) we obtain

$$t = \int \frac{c\,d\tau}{\sqrt{1 - \sum_i \mu_i \sin^2(\nu\tau + \varphi_i^0)}}, \tag{4.20}$$

where c, μ_i are constants, and $\sum_i |\mu_i| < 1$. This integral is known to be elliptic, while $\sin(\nu\tau)$, $\cos(\nu\tau)$, as well as radical in (4.20), are elliptic functions of t. Thus, by (4.17), $\Omega_{12}, \dots, \Omega_{1n}$ are single-valued functions of the complex time t.

It is interesting to note that in the most general case the square of Ω_{12}, as well as those of $\Omega_{1,s+2}$, $\Omega_{2,s+2}$ $(1 \leqslant s \leqslant n-2)$, is an entire function of the new time τ. This follows from (4.17) and the equation

$$(I_1 + I_2)\frac{d}{d\tau}\Omega_{12}^2 = 2(I_2 - I_1)\sum_{k=3}^{n}\Omega_{1k}\Omega_{2k}\,.$$

It may happen that the EPS equations for the group $SO(n)$ may also be integrable in the more general case when, considered in certain frame, the mass tensor is of diagonal form and the constraints are defined by any combination of equalities $\Omega_{ij} = 0$.

As it was mentioned above, in EPS equations the kinetic energy (metric) and the constraints are left-invariant on a Lie group. This gives the possibility to obtain a closed system on the corresponding Lie algebra.

Now consider another interesting case of the Euler–Poincaré equations on Lie groups when the kinetic energy is left-invariant, whereas the constraints are *right-invariant*. These are so called L–R systems, studied by Veselov and Veselova [23]. The right invariance condition means that in coordinates in the Lie algebra g (in quasivelocities) a constraint is determined by the relation

$$\langle \Omega, \mathcal{N} \rangle = \text{const}, \qquad \Omega \in g\,,\ \mathcal{N} \in g^*\,,$$

where, in contrast to the case of left-invariant constraints, $\mathrm{Ad}_x^* \mathcal{N} = \text{const}$.

EXAMPLE 6. The most descriptive illustration of an L–R system is the Veselova problem concerning the rotations of a rigid body with a fixed point under the action of the following nonholonomic constraint

$$(\omega, \gamma) = q\,, \qquad q = \text{const} \tag{4.21}$$

(the projection of the angular velocity vector $\omega \in \mathbb{R}^3$ to some unit vector γ *fixed in space* is constant) [9, 23]. Here $\mathfrak{G} = SO(3)$ and the Euler–Poincaré equations (4.4) along with the kinematic equations get the form

$$J\dot{\omega} = J\omega \times \omega + \lambda\gamma\,, \qquad \dot{\gamma} = \gamma \times \omega\,, \tag{4.22}$$

where, as before, J is the inertia tensor relative to the fixed point. Using this and differentiating (4.21), we find

$$\lambda = -\frac{(J\omega \times \omega\,,\, J^{-1}\gamma)}{(\gamma\,,\, J^{-1}\gamma)}\,.$$

Note that the system (4.22) admits the following representation

$$\begin{aligned} &\dot{Q} = Q \times \omega\,, && \dot{\gamma} = \gamma \times \omega\,, \\ &Q = J\omega - (J^0\omega\,,\,\gamma)\gamma\,, && J^0 = J - E\,, \end{aligned} \tag{4.23}$$

E being the unit matrix. Hence the vector Q, similarly to γ, is thus a fixed vector in space. Therefore, the system possesses three independent integrals

$$(Q\,,\,Q)\,,\quad (Q\,,\,\gamma)=(\omega\,,\,\gamma)\,,\quad (\gamma\,,\,\gamma)=1\,. \tag{4.24}$$

The second integral coincides with the constraint equation. Since under the condition $q \neq 0$ the constraint (4.21) is nonstationary, the kinetic energy of the body cannot be constant. However, instead of an energy integral, there exists the following analog of the Jacobi–Painlevé integral

$$(J\omega\,,\,\omega)-2(\omega\,,\,\gamma)(J^0\omega\,,\,\gamma)\,. \tag{4.25}$$

Finally, the system considered in the phase space $(\omega\,,\,\gamma)$ has the integral invariant with density $\sqrt{(J^{-1}\gamma\,,\,\gamma)}$ and, therefore, it is integrable by the classical Jacòbi theorem.

In addition to the multi-dimensional generalization of the Suslov problem, it is worthwhile considering the generalization of the equations (4.22) which is an L–R system on the group $SO(n)$. In this case it is natural to take constraint equations in form (4.9) with the only difference that Ω_c must be replaced by the angular velocity in the space Ω_s. For the sake of generality, the zeros at the corresponding entries can be replaced by arbitrary constants:

$$\Omega_s = R\Omega_c R^{-1} = \begin{pmatrix} 0 & \Omega_{12} & \dots & \Omega_{1n} \\ -\Omega_{12} & & & \\ \vdots & & \widetilde{\Omega} & \\ -\Omega_{1n} & & & \end{pmatrix}, \tag{4.26}$$

$$\widetilde{\Omega}_{ij} = \text{const}\,, \qquad 2 \leqslant i < j \leqslant n\,.$$

Let $\{e_1\,,\,\dots\,,\,e_n\}$ be a fixed orthonormal basis in $\mathbb{R}^n$, $(e_{i1}\,,\,\dots\,,\,e_{in})$ be the projections of e_i to the axes of an orthonormal frame attached to the body, so that the rotation matrix is written in the form $R = (e_1 \dots e_n)^T$. Define 2-vectors $\mathcal{E}^{(rs)} = e_r \wedge e_s \in \bigwedge^2 \mathbb{R}^n$, $r\,,\,s = 1\,,\,\dots n$ represented by the $n \times n$ skew symmetric matrices $\mathcal{E}^{(rs)}_{ij} = e_{ri}e_{sj} - e_{rj}e_{si}$. These 2-vectors give an orthonormal basis in $so(n)$ with respect to the Killing form $\langle\cdot\,,\,\cdot\rangle$. Therefore, (4.26) is equivalent to the constraints

$$\langle\Omega\,,\,\mathcal{E}^{(rs)}\rangle = \widetilde{\Omega}_{ij}\,, \qquad 2 \leqslant r < s \leqslant n\,, \tag{4.27}$$

where the $\mathcal{E}^{(rs)}$'s can be regarded as generalizations of the vector γ in (4.21). The corresponding Euler–Poincaré equations with multipliers, as well as the kinematic equations, can be represented in the following matrix form

$$\dot{M} + [\Omega\,,\,M] = \sum_{\substack{(rs)\\ 2\leqslant r<s\leqslant n}} \lambda_{(rs)}\mathcal{E}^{(rs)}\,, \qquad \dot{\mathcal{E}}^{(rs)} + [\Omega\,,\,\mathcal{E}^{(rs)}] = 0\,. \tag{4.28}$$

As above, $M = \mathcal{A}\Omega$ is the angular momentum of the body with the inertia tensor $\mathcal{A}$. Differentiating (4.27) and using (4.28), we obtain the following system of linear equations for determining the multipliers $\lambda_{(rs)}$

$$\sum_{\substack{(kl)\\ 2\leqslant k<l\leqslant n}} \langle\mathcal{A}^{-1}\mathcal{E}^{(kl)}\,,\,\mathcal{E}^{(rs)}\rangle\lambda_{(k\,,\,l)} = \langle\mathcal{E}^{(rs)}\,,\,\mathcal{A}^{-1}[\Omega\,,\,M]\rangle\,,\ 2 \leqslant r < s \leqslant n\,. \tag{4.29}$$

Now let $H = \operatorname{span}(e_1 \wedge e_2, \ldots, e_1 \wedge e_n)$ and H' be the orthogonal complement to H in $\bigwedge^2 \mathbb{R}^n$. It turns out that a part of the system (4.28) is separated and can be written in the following commutative form

$$\text{(4.30)} \qquad \begin{gathered} \dot{Q} = [Q, \Omega], \qquad \dot{e}_1 = -\Omega e_1, \\ Q = M|_H + \Omega|_{H'} = \Omega + (M - \Omega)\Gamma - \Gamma(M - \Omega), \qquad \Gamma_{ij} = e_{1i}e_{1j}, \end{gathered}$$

where $M|_H$, $\Omega|_{H'}$ are projections on the linear subspaces H, $H' \in \bigwedge^2 \mathbb{R}^n$. Equations (4.30) represent a closed system for determining Ω and e_1 as functions of time t. This system possesses multi-dimensional analogs of the integrals (4.24)

$$\begin{gathered} \langle Q, Q\rangle = -\frac{1}{2}\operatorname{tr}(M\Gamma + \Gamma M)^2, \qquad (e_1, e_1), \\ Q_{ij}e_{1k} + Q_{jk}e_{1i} + Q_{ki}e_{1j} = \Omega_{ij}e_{1k} + \Omega_{jk}e_{1i} + \Omega_{ki}e_{1j}, \qquad 1 \leqslant i < j < k \leqslant n. \end{gathered}$$

Besides, it has an analog of the Jacobi–Painlevé integral (4.25), as well as the integral invariant. In the variables (Ω_{ij}, e_{1i}) the latter has the density $\mu = \sqrt{\det \mathcal{A}|_H}$, where $\mathcal{A}|_H$ denotes the restriction of the operator $\mathcal{A}$ to the linear space H.

Similarly to the multi-dimensional generalization of the Chaplygin problem discussed in §2, the system (4.28) or (4.30) also may be integrable.

§5. Historical comments

The reader, familiar with studies on the dynamics of multi-dimensional rigid bodies, may have already noticed that our references regarding the origins of the basic concepts of the theory seem quite unusual. In this connection, the authors would like to give some explanations.

In the current literature devoted to the formulation and the equations of the multi-dimensional analog of the Euler problem, the reader is usually referred to Arnold's well-known paper [1], in which it is also shown that these equations are Hamiltonian on the orbits of the coadjoint action of the group $SO(n)$. However, as it was mentioned above, more than a century ago, the idea of such a generalization was put forward by Cayley [6], and in 1873 Frahm [10] obtained dynamical and kinematic equations for the problem in explicit form. The latter also found a complete set of trivial integrals which are analogs of the classical angular momentum integral and the "area" integral. Moreover, for $n = 4$, he derived a condition on the coefficients of the inertia tensor for the dynamical equations to have an additional quadratic integral and, thereby, to be integrable "by quadratures". This condition, in fact, coincides with the general relation (2.13). (At present, condition (2.13) has become associated with the well-known Manakov paper [16]).

All this enables us to call the equations of free motion of the n-dimensional top the *Euler–Frahm equations.*

After Frahm, this dynamical system was discovered by H. Weyl [25], 1923; Blaschke [3], 1942 (of course, this list can be scarcely regarded as complete). Some authors tried in vain to perform the explicit integration of the Euler–Frahm equations in the four-dimensional case without knowing that the scheme of such an integration procedure had been given by Schottky as early as in 1891 [21]. The latter had noticed the remarkable connection (in fact an isomorphism) between the integrable cases found by Frahm and Clebsch's second integrable case of the Kirchhoff equations describing rigid body motion in an unbound volume of ideal liquid. (Both are six-dimensional dynamical systems). Explicit integration of the latter case had already been performed by that time by the brilliant analyst Kötter (see [15]). As Schottky asserted, Kötter's method of integration could be applied (almost without modifications) to Frahm's case. Later the Schottky paper was forgotten, and only in 1986 the explicit formulas describing the isomorphism mentioned above were obtained again by Bobenko [4], while the generalized Euler equations for the four-dimensional top, as well as the n-dimensional top, were integrated by using the recently discovered finite-gap integration method.

Generally, the problem of explicit integration of dynamical systems in theta-functions of time as a complex argument was very popular in Germany at the end of the last century.

According to some publications, the equations for geodesics of left-invariant metrics on Lie groups, regarded as natural generalizations of the Euler equations, came into use quite recently. However, the real history is different. As early as in 1901, Poincaré represented the Lagrange equations in "group" variables. He gave special attention to the case when the Lagrangian is a left-invariant function and mentioned that half of the equations form a closed system defined on the corresponding Lie algebra. As an example, he considered precisely the Euler equations for the n-dimensional top.

It is interesting to note that this paper of Poincaré's is well known to the physics community and is practically unknown to mathematicians dealing with the theory of *Euler–Poincaré equations on Lie algebras*!

It is also worth mentioning that physicists were definitely aware of the group structure involved in many dynamical systems. For instance, in some publications it is accepted that the interpretation of the Kirchhoff equations as the Euler–Poincaré equations on the Lie algebra $e(3)$ was first given by Novikov and Schmeltzer in [19]. But, as a matter of fact, the group structure of the Kirchhoff equations had been already realized and presented in explicit form by Birkhoff and Braquell in 1945 (see [2, 5]). Besides, even earlier, some researchers from the German applied mechanics community had used the so-called *Motor calculus* (*Motorrechnung*), which is, in fact, a matrix realization of the Lie algebra $e(3)$. The main object of the theory is represented by a second-order tensor (*Impulsmotor*), which turns out to be a skew-symmetric analog of the kinetic momentum relative to the group $E(3)$ defined in (1.7) (see [18]).

The generalization of the Euler–Poincaré equations to systems with left-invariant constraints (Euler–Poincaré–Suslov equations) seems to have been first considered in the paper [13] dealing with the existence of an invariant measure, while the same problem for systems with right-invariant constraints, as well as other relevant questions, were discussed in [9, 23].

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