*Sbornik: Mathematics* **187**:5 679–684 
*Matematicheskiĭ Sbornik* **187**:5 59–64 UDC 517.9+531.01

# Topology of domains of possible motions of integrable systems

V. V. Kozlov and V. V. Ten

**Abstract.** A study is made of analytic invertible systems with two degrees of freedom on a fixed three-dimensional manifold of a level of the energy integral. It is assumed that the manifold in question is compact and has no singular points (equilibria of the initial system). The natural projection of the energy manifold onto the two-dimensional configuration space is called the domain of possible motions. In the orientable case it is a sphere with $k$ holes and $p$ attached handles. It is well known that for $k = 0$ and $p \geqslant 2$ the system possesses no non-constant analytic integrals on the corresponding level of the energy integral. The situation in the case of domains of possible motions with a boundary turns out to be very different. The main result can be stated as follows: there are examples of analytically integrable systems with arbitrary values of $p$ and $k \geqslant 1$.

Bibliography: 10 titles.

## § 1. Introduction. Main result

We consider an invertible dynamical system with two degrees of freedom. Let $M$ be the configuration space of the system: $M$ is a two-dimensional manifold. The phase space is four-dimensional; it is the space of the cotangent bundle $T^*M$.

The Hamiltonian function $H\colon T^*M \to \mathbb{R}$ of an invertible system has the form

$$H = T + V, \tag{1.1}$$

where

$$T = \frac{1}{2}\sum g_{ij}(x)y_iy_j$$

is the kinetic energy (the metric on $M$ is Riemannian), that is, a positive-definite quadratic form in the momenta $y_1$ and $y_2$, and $V\colon M \to \mathbb{R}$ is the potential energy (a function of the coordinates $x_1$, $x_2$ on $M$). In what follows we assume that $M$ is endowed with the structure of an oriented analytic manifold and $H$ is an analytic function on $T^*M$. For a non-orientable $M$, there is a natural

This research was carried out with the financial support of the Russian Foundation for Fundamental Research (grant no. 93-013-16244) and of the International Scientific Foundation (grant no. MSU 000).

*AMS* 1991 *Mathematics Subject Classification.* Primary 58F05; Secondary 70G25, 70F10.

two-sheeted covering $N \to M$, where $N$ is oriented and the dynamical system on $T^*M$ can be 'lifted' to an invertible system on $T^*N$.

The Hamilton equations

$$\dot{x}_k = \frac{\partial H}{\partial y_k}, \quad \dot{y}_k = -\frac{\partial H}{\partial x_k} \qquad (k = 1, 2) \tag{1.2}$$

have an energy integral given by (1.1). We choose a fixed constant value of the energy, which we denote by $h$. The three-dimensional level surface of the energy integral

$$\Sigma_h = \{x, y : H = h\}$$

is a regular analytic manifold if $\Sigma_h$ contains no critical points of $H$, that is, equilibrium states of the system under study. This assumption is clearly equivalent to the requirement that $h$ is not a critical value of the potential energy $V$. We shall consider the case when $\Sigma$ is compact.

The natural projection $T^*M \to M$ (sending $(x, y)$ to $x$) takes $\Sigma_h$ to a compact domain (possibly with boundary) $B_h$ on $M$. This domain is said to be the *domain of possible motions*. Since $T \geqslant 0$, we have

$$B_h = \{x \in M : V(x) \leqslant h\}.$$

The geometry and dynamics of the domains of possible motions were discussed in [1].

We now assume that the system (1.2) on $\Sigma$ admits a non-constant analytic integral

$$F: \Sigma \to \mathbb{R}.$$

By the Liouville theorem, non-critical level surfaces of $F$ are the union of two-dimensional tori with conditionally periodic trajectories. An analytic function on a compact manifold has finitely many critical values and, therefore, $\Sigma$ can be decomposed into finitely many pieces with a 'regular' fibering into two-dimensional tori.

It turns out that the complete integrability of the system (1.2) on $\Sigma$ depends strongly on the topology of the domain of possible motions $B$. The first result in this context was obtained in [2] and can be stated as follows: if $\partial B = \varnothing$ ($M$ compact and $h > \max V$) and the genus $p$ of the surface $B = M$ (the number of handles) is $\geqslant 2$, then the dynamical system on $\Sigma$ has no non-constant analytic integrals. On the other hand, there are many examples of integrable problems for $p = 0$ (sphere) and $p = 1$ (torus). New proofs of this result based on different ideas were later obtained in [3]–[6].

We shall now consider the case when the boundary $\partial B$ is non-empty. The topological classification of such domains is well known. They are spheres with $k \geqslant 1$ holes and $p \geqslant 0$ handles. The number $p$ is called the *genus* of $B$. Our main result can be stated as follows:

**Theorem 1.** *Given any $k \geqslant 1$ and $p \geqslant 2$, we can find an analytic system* (1.2) *and a value $h$ of the energy such that*

1. *the restriction of* (1.2) *to $\Sigma_h$ admits a non-constant analytic integral and*
2. *the compact domain $B_h$ is homeomorphic to a sphere with $k$ holes and $p$ handles.*

Hence, the domain of possible motions with boundary for an analytically integrable system with two degrees of freedom can have an arbitrarily complex topological structure.

It may be instructive in this context to compare Theorem 1 with the following result of [7]. Let $V \equiv 0$ and suppose that there is a two-dimensional manifold with boundary $M_0 \subset M$ such that $\chi(M_0) < 0$ and the boundary of $M_0$ is geodesically convex. The equations (1.2) then have no additional analytic integrals. For example, if $M_0$ is a disc with $k \geqslant 2$ holes, then $\chi(M_0) = 1 - k < 0$.

## § 2. The case $p \leqslant 1$

It is very simple to prove Theorem 1 in the cases $p = 0$ and $p = 1$. We produce systems that are analytically integrable on $\Sigma$ and are such that the projections of $\Sigma$ on $M$ (domains of possible motions) contain any given number of holes.

We take first the case $p = 0$. Let $M = \mathbb{R}^2 = \{x_1, x_2\}$ and assume that the kinetic energy is generated by the planar metric

$$T = (y_1^2 + y_2^2)/2 \tag{2.1}$$

and that the potential $V$ is given by the sum

$$f(x_1) + g(x_2), \tag{2.2}$$
$$f(z) = T_4(z) = 8z^4 - 8z^2 + 1, \qquad g(z) = T_{2k+2}(z).$$

Here, the $T_n(z) = \cos(n \arccos z)$ are the Chebyshev polynomials.

It is obvious that $V(x_1, x_2) \to +\infty$ as $x_1^2 + x_2^2 \to \infty$. The potential energy has exactly $k$ local maxima where its value is always 2. We set $h = 2 - \varepsilon$ where $\varepsilon$ is a small positive number. It is then self-evident that $B_h$ is homeomorphic to a disc with $k$ holes.

It remains to note that an invertible system with the kinetic energy (2.1) and a potential given by (2.2) is completely integrable. It admits the additional analytic integral

$$F = \frac{y_1^2}{2} + f(x_1),$$

which is independent of the energy integral $H = T + V$.

For $p = 1$ and any $k \geqslant 0$, we can also specify an invertible system that is integrable by separation of variables. Let

$$M = \mathbf{T}^2 = \{x_1, x_2 \bmod 2\pi\},$$
$$T = \frac{y_1^2 + y_2^2}{2}, \qquad V = \cos x_1 + \cos(k+1)x_2.$$

It is easy to confirm that the domain $B_h$ for $h = 2 - \varepsilon$, where $\varepsilon > 0$ is small, is homeomorphic to a disc with a single handle and $k$ holes. The additional analytic integral is given, for example, by

$$F = \frac{y_1^2}{2} + \cos x_1.$$

## § 3. The case $p > 1$

The construction of an analytically integrable system with domain of possible motions of genus $p > 1$ is based on the idea developed in § 2.

On the Euclidean plane $\mathbb{R}^2 = \{x_1, x_2\}$, we consider the square

$$\Pi = [-\pi, \pi] \times [-\pi, \pi]$$

and define the kinetic and potential energies by

$$T = \frac{y_1^2 + y_2^2}{2}, \quad V = -\cos x_1 - \cos x_2; \qquad y_i = \dot{x}_i. \tag{3.1}$$

For $h = 1$ the domain of possible motions $B_h \subset \Pi$ is a curvilinear cross intersecting the boundary $\partial\Pi$ along the segments

$$a_1 = \{-\pi\} \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \qquad a_2 = \{\pi\} \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$
$$b_1 = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \{-\pi\}, \qquad b_2 = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \{\pi\}.$$

Let

$$K = \left\{(x_1, x_2) \in \Pi : V(x_1, x_2) \leqslant \frac{3}{2}\right\} \tag{3.2}$$

be a larger cross bounded by segments $\alpha_1 \supset a_1$, $\alpha_2 \supset a_2$, $\beta_1 \supset b_1$, and $\beta_2 \supset b_2$.

We now consider $2p + k - 2$ copies of the curvilinear cross (3.2). We shall denote the Cartesian coordinates in the $i$th cross $K_i$ by $x_1^i$ and $x_2^i$. The kinetic and potential energies in $K_i$ will be defined by (3.1) with $x_k$ and $y_k$ replaced by $x_k^i$ and $y_k^i$. We denote the boundary segments of $K_i$ by $\alpha_1^i$, $\alpha_2^i$, $\beta_1^i$, $\beta_2^i$. Two crosses $K_i$ and $K_j$ can be glued together, for example, along the segments $\alpha_1^i$ and $\alpha_2^j$ by identifying the points with coordinates $(-\pi, x_2^i)$ and $(\pi, x_2^j)$. Such gluing will be denoted by the symbol $(\alpha_1^i, \alpha_2^j)$. The gluing can be carried out in such a way that it gives rise to an analytic two-dimensional manifold (with boundary). Similarly, we can define the gluing $(\beta^i, \beta^j)$. We now glue together $K_1, \dots, K_{2p+k-2}$ according to the formula

$$(\beta_1^1, \beta_2^1) + (\alpha_1^1, \alpha_1^2) + (\alpha_2^1, \alpha_2^2) + \cdots + (\beta_1^{2p-2}, \beta_1^{2p-1}) + (\beta_2^{2p-2}, \beta_2^{2p-1})$$
$$+ \sum_{i=2}^{k} \left((\alpha_1^{2p+i-3}, \alpha_2^{2p+i-2}) + (\beta_1^{2p+i-2}, \beta_2^{2p+i-2})\right) + (\alpha_1^{2p+k-2}, \alpha_2^{2p-1}). \tag{3.3}$$

Here, "+" denotes successive gluing operations. As a result, we obtain an analytic compact manifold with boundary $M$.

**Lemma.** *The manifold $M$ is homeomorphic to a sphere with $k$ holes and $p$ attached handles.*

We shall prove this assertion later.

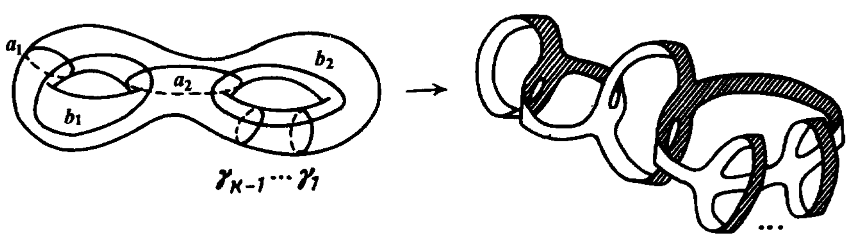
The potential energy $V\colon M \to \mathbb{R}$ is an analytic function by periodicity. The domain of possible motions

$$B_1 = \{V \leqslant 1\} \subset M$$

is clearly homeomorphic to $M$. We claim that the invertible mechanical system thus constructed is analytically integrable. In fact, we can define a function $F$ on $T^*K_i$ ($i = 1, \dots, 2p + k - 2$) by the formula

$$\frac{(y_1^i)^2}{2} - \cos x_1^i.$$

This is clearly an integral of the invertible Hamiltonian system in question. The function $F$ is defined almost everywhere on $T^*M$ and, by the construction of $M$, it can be extended by continuity to an analytic function on the whole phase space $T^*M$. This proves the theorem.




*Proof of the Lemma.* Let us consider a closed surface of genus $p$ (illustrated in the figure for $p = 2$) and $2p$ 'canonical' closed cycles $a_1, b_1, \dots, a_p, b_p$; let $\gamma_1, \dots, \gamma_{k-1}$ be homological cycles intersecting $b_p$ (see the figure). Cutting the surface along these $2p + k - 1$ cycles, we obtain $k$ pieces each of which is homeomorphic to a disc. Let us surround each cycle by a thin band. The domain on the surface outside such bands is a disconnected union of $k$ discs. Hence the union of all the bands (see the figure) is a surface with boundary of genus $p$ whose boundary consists of exactly $k \geqslant 1$ connected components. It is easy to confirm that it can be obtained by gluing together $2p + k - 2$ crosses in accordance with (3.3). This proves the lemma.

## § 4. Concluding remarks

**1.** A function $F\colon T^*M \to \mathbb{R}$ is said to be a *conditional integral* (in the Birkhoff sense) of the Hamiltonian system (1.2) if $\dot{F} = 0$ on some surface $\Sigma_h = \{H = h\}$. Theorem 1 can be formulated more precisely as follows:

**Theorem 2.** *Given any compact surface with boundary $N$, there are an analytic invertible Hamiltonian system and a value $h$ of the total energy such that*

(1) *$B_h$ is homeomorphic to $N$,*

(2) *there is a non-trivial conditional integral that is a quadratic polynomial in the momenta.*

Hence, there are no topological obstructions to the existence of conditional quadratic integrals for domains of possible motions with boundary. It is interesting to note that in the case of conditional integrals that are linear in the momenta such obstructions do exist. It was shown in [9] that the existence of a conditional linear integral for $H = h$ implies that the Euler characteristic of $B_h$ is non-negative.

**2.** It would be interesting to investigate obstructions to the existence of conditional polynomial integrals for non-invertible systems with gyroscopic forces when $\partial B \neq \varnothing$. This problem was studied in [10] in the case when $B$ coincides with the whole $M$. If $M$ has genus greater than 1, then there are no polynomial conditional integrals. If $M$ is homeomorphic to a torus, then a necessary condition for Birkhoff integrability can be stated as follows: the integral of the 2-form of the gyroscopic forces over the torus is zero.

**3.** The problem of topological obstructions to the complete integrability for systems with many degrees of freedom in the case $\partial B \neq \varnothing$ remains open at present. In this context, one would have to consider the conditions for the existence of $n$ involutive integrals, where $n$ is the number of degrees of freedom of the system.

## Bibliography

[1] V. V. Kozlov, "Global calculus of variations and classical mechanics", *Uspekhi Mat. Nauk* **40**:2 (1985), 33–60; English transl. in *Russian Math. Surveys* **40**:2 (1985).

[2] V. V. Kozlov, "Topological obstructions to the integrability of natural mechanical systems", *Dokl. Akad. Nauk SSSR* **249** (1979), 1299–1302; English transl. in *Soviet Math. Dokl.* **20** (1979).

[3] V. N. Kolokol'tsov, "Geodesic flows on two-dimensional manifolds with an additional flow integral that is polynomial with respect to velocities", *Izv. Akad. Nauk SSSR Ser. Mat.* **46** (1982), 994–1010; English transl. in *Math. USSR-Izv.* **21** (1983).

[4] I. A. Taĭmanov, "Topological obstructions to the integrability of geodesic flows on non-simply connected manifolds", *Izv. Akad. Nauk SSSR Ser. Mat.* **51** (1987), 429–435; English transl. in *Math. USSR-Izv.* **30** (1988).

[5] S. V. Bolotin, "Doubly asymptotic trajectories of minimal geodesics", *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **1992**:1, 92–96; English transl. in *Moscow Univ. Math. Bull.* **47**:1 (1992).

[6] G. P. Paternain, "On the topology of manifolds with completely integrable geodesic flows", *Ergodic Theory Dynamical Systems* **12** (1992), 109–121.

[7] S. V. Bolotin, "Non-integrability of the problem of $n$ centers for $n > 2$", *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **1984**:3, 65–68; English transl. in *Moscow Univ. Math. Bull.* **39**:3 (1984).

[8] V. V. Kozlov, "Integrability and non-integrability in Hamiltonian mechanics", *Uspekhi Mat. Nauk* **38**:1 (1983), 3–67; English transl. in *Russian Math. Surveys* **38**:1 (1983).

[9] D. L. Abarov, "Topological obstructions to the existence of conditionally linear integrals", *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **1984**:6, 72–75; English transl. in *Moscow Univ. Math. Bull.* **39**:6 (1984).

[10] S. V. Bolotin, "First integrals of systems with gyroscopic forces", *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **1984**:6, 75–82; English transl. in *Moscow Univ. Math. Bull.* **39**:6 (1984).

Moscow State University

Received 3/OCT/95
Translated by D. MATHON

Typeset by $\mathcal{A}\!\mathcal{M}\!\mathcal{S}$-TEX