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MATHEMATICAL PHYSICS

Nonintegrability of a System of Interacting Particles with the Dyson Potential

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1. The dynamics of a system of n interacting particles of identical mass is described by a Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2}\sum y_i^2 + \sum_{i<j} V(x_i - x_j). \quad (1)$$

Here $x_1, x_2, \ldots, x_n$ are the coordinates of the particles; $y_1, y_2, \ldots, y_n$ are their momenta; and V is the potential energy of interaction. Following Dyson [1], we consider the case

$$V(z) = \ln|\sin z|. \quad (2)$$

Systems with this potential were examined in [1], where the statistical properties of the energy levels of a one-dimensional classical Coulomb gas were analyzed. As in the situation noted by Calogero for a system of point vortices in the plane [2], the equilibrium positions of system (1), (2) determine stationary collinear configurations on the sphere (point vortices are located in the equatorial plane that uniformly rotates about an axis also lying in this plane).

Since the function V is 2π-periodic (and even π-periodic), we can assume that the particles move in circles. The system with Hamiltonian (1) always admits two integrals

$$H, \quad F = \sum y_i.$$

The conditions on V under which the system in question is completely integrable (admits a set of n independent integrals that are polynomials in $y_1, y_2, \ldots, y_n$) were sought in many studies (see reviews in [2, 7]). If V is a nonconstant analytic periodic function without singularities, then the system with Hamiltonian (1) cannot be completely integrable for $n \geq 3$ [3, 4]. The Dyson potential (2) has a real logarithmic singularity. The integrability of this system was discussed in [5].

Dyson noted that the system with potential (2) admits the family of equilibriums

$$x_j^0 = x_0 + \frac{\pi j}{n}, \quad j = 1, 2, \ldots, n, \quad x_0 \in \mathbb{R}. \quad (3)$$

The frequencies of small-amplitude oscillations were calculated in [6] to be

$$\omega_s^2 = 2s(n-s), \quad s = 1, 2, \ldots, n. \quad (4)$$

The equality $\omega_n = 0$ is associated with the fact that equilibriums (3) are nonisolated.

2. Consider the simplest nontrivial case $n = 3$. Using the momentum integral F, it is possible to reduce the number of degrees of freedom by one. To do this, we proceed to a noninertial barycentric frame of reference using the canonical transformation $x, y \to q, p$:

$$y_1 = p_1 + p_3, \; y_2 = -p_1 + p_2 + p_3, \; y_3 = -p_2 + p_3,$$
$$q_1 = x_1 - x_2, \; q_2 = x_2 - x_3, \; q_3 = x_1 + x_2 + x_3.$$

By taking into account the equality $p_3 = 0$ and the evenness of the potential, the Hamiltonian of the reduced system is given by

$$H = p_1^2 - p_1 p_2 + p_2^2 + V(q_1) + V(q_2) + V(q_1 + q_2). \quad (5)$$

This system has the stable equilibrium $q_1 = q_2 = \frac{\pi}{3}$ with equal frequencies of small oscillations $\omega_1 = \omega_2 = 2$ [according to (4)]. Subtracting $\ln\frac{\sqrt{3}}{2}$ from the potential, we can assume that the total energy in the equilibrium state is zero.

It is reasonable to expect the system with potential (2) to be integrable for small positive values of the total energy h. The situation is the same as in the well-known Hénon–Heiles system (see [6] and also [7]). Applying the normal form method and taking into account the resonance $\omega_1 = \omega_2$, it is possible to find a quasi-integral that varies very slowly with time in a neighborhood of the equilibrium point (for the Hénon–Heiles system, such a function was calculated by Gustavson [8]).

3. Numerical computations have confirmed this assumption. Figure 1a shows the almost integrable behavior of the system for $H = 10$. When H is large, the behavior of the system becomes chaotic in the neighborhood of the separatrices. The formal series defining the quasi-integral diverge, and the quasi-integral does not approximate the behavior of the system for large H. Figures 1b and 1c correspond to the energies $H = 20$ and $H = 22$, respectively, for which the behavior of the system becomes stochastic.

4. It is possible to analyze the integrability of the system of interacting particles with the Dyson potential

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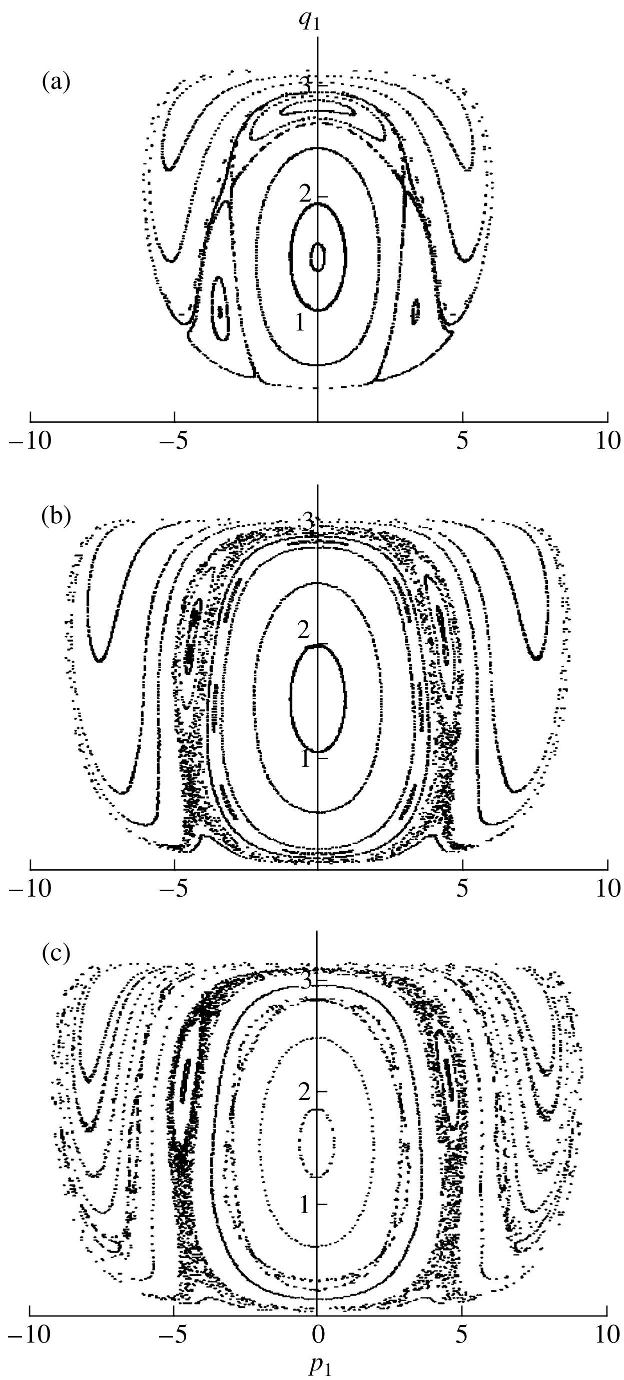

Fig. 1. Poincare section for the energy H = (a) 10, (b) 20, and (c) 22.

from a simpler point of view, assuming that the canonical coordinates x and y and time t are complex variables. The first integrals are then sought in the form of polynomials in momenta with single-valued analytic coefficients [3]. Because of a logarithmic singularity of the potential, the energy H branches in the complex phase space, while the function F is, of course, single-valued.

It turns out that the momentum integral F is a unique polynomial integral with single-valued coefficients in the Dyson system. This statement can be proved using the results of [9].

Indeed, let $F_j = y_j$, $1 \le j \le n$, be a complete set of independent integrals in the problem of circular motion of noninteracting particles. By virtue of the Hamiltonian system with Hamiltonian (1), (2), the derivatives of these functions can be calculated to be

$$\dot{F}_j = \sum_{k=1}^{n} \cot(x_j - x_k), \quad 1 \le j \le k. \tag{6}$$

Substituting any solution of the "free" system, for example,

$$x_1 = \frac{\pi}{n}, \ldots, x_{n-1} = \frac{(n-1)\pi}{n}, \quad x_n = t$$

into the right-hand sides of (6), we find that they are meromorphic functions in the plane of complex time, and $n - 1$ points $t = \frac{\pi}{n}, \ldots, t = \frac{(n-1)\pi}{n}$ are simple poles. Calculating the residues of $(\dot{F})_t$ at these points, it is easy to see that they (as vectors in $\mathbb{C}^n$) are linearly independent. Hence, according to [9], the system in question has only one single-valued polynomial first integral.

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