Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021–8928(02)00132-6

J. Appl. Maths Mechs, Vol. 66, No. 6, pp. 897–904, 2002

0021–8928/02/$—see front matter

STEADY MOTIONS OF A CONTINUOUS MEDIUM, RESONANCES AND LAGRANGIAN TURBULENCE†

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(*Received* 22 *August* 2001)

A method which enables one to establish a non-regularity property of the motion of fluid particles (known as chaotic advection or Lagrangian turbulence) for typical steady flows is developed. The method is based on expanding solutions of the equations of motion of a continuous medium in powers of a small parameter and using the conditions for the destruction of invariant resonant tori when perturbations are added. It is shown that the velocity field, defined as the solution of the Burgers equations, generates a generally non-regular dynamical system. For an ideal barotropic fluid in an irrotational force field, the method proposed yields a well-known necessary condition for chaotization: the velocity field is collinear with its curl. Special attention is given to investigating the chaotization of typical steady flows of a heat-conducting perfect gas. © 2003 Elsevier Science Ltd. All rights reserved.

1. REGULAR AND CHAOTIC STEADY FLOWS

Consider the steady flow of a fluid in a domain D of a three-dimensional Euclidean space $\mathbb{R}^3 = \{x, y, z\}$. In what follows we shall consider mainly flows which are 2π-periodic in the coordinates x and y; in that case the domain D will be a direct product $\mathbb{T}^2 \times \mathbb{R}$. where $\mathbb{T}^2 = \{x, y \bmod 2\pi\}$ is a two-dimensional torus. The steady velocity field $\mathbf{v}$ is found as a solution, not explicitly dependent on the time t, of the equations of motion of a continuous medium. Examples of specific models of continuous media will be considered below.

The motion of the fluid particles is described by an autonomous dynamical system

$$d\mathbf{x}/dt = \mathbf{v}(\mathbf{x}), \mathbf{x} \in D. \tag{1.1}$$

Let g^t be the phase flow. The action of the group g^t on the domain D is *advection* (the transport of the fluid particles). Chaotic advection is known as *Lagrangian turbulence*.

A rigorous study of Lagrangian turbulence presupposes the availability of a rigorous definition of chaoticity of dynamical system (1.1). However, for a variety of reasons, it is impossible to formulate a universal property of chaoticity. Chaoticity is naturally contrasted with the regular behaviour of system (1.1). There are various non-equivalent approaches to defining a regular dynamical system (see, e.g. [1]). Common to them is the existence of non-trivial tensor invariants. The simplest of them are integrals, fields of symmetries and integral invariants.

We recall that a locally non-constant function $f: D \to \mathbb{R}$ is called an *integral* of system (1.1) if

$$df/dt = (\partial f/\partial \mathbf{x}, \mathbf{v}) = 0$$

The flow domain is stratified into invariant integral surfaces $N_c = \{\mathbf{x} \in D: f(\mathbf{x}) = c\}$. If a surface N_c is compact (i.e. bounded) and the velocity $\mathbf{v}$ nowhere vanishes on it, the surface is topologically equivalent to a two-dimensional torus.

A field $\mathbf{w}$ is called a *field of symmetries* if $[\mathbf{v}, \mathbf{w}] = 0$, where $[\cdot, \cdot]$ is the Jacobi bracket. We are interested in non-trivial fields of symmetries, when the vectors $\mathbf{v}$ and $\mathbf{w}$ are not collinear. The phase flow of a field $\mathbf{w}$ transforms solutions of system (1.1) into solutions of the same system. A more general concept is that of a frozen-in field of directions $\mathbf{w}$, defined by the condition $[\mathbf{v}, \mathbf{w}] = \lambda\mathbf{w}$, where λ is some function defined in the domain D. A flow g^t transforms the family of integral curves of the field $\mathbf{w}$ into itself (see [2, 3]). The symmetry properties of fluid flow play an important role in the theory of turbulence [4].

†*Prikl. Mat. Mekh.* Vol. 66, No. 6, pp. 939–947, 2002.

A differential 1-form ω generates an *integral invariant* if

$$I(t) = \int_{g^t\gamma} \omega = \text{const}$$

for all closed cycles γ. Of course, one must also assume that $d\omega \neq 0$ (ω does not reduce to a total differential). Otherwise, $I(t) \equiv$ const.

An instructive example of a regular flow is the steady flow of a barotropic viscous fluid in an irrotational force field. It admits of an integral (the Bernoulli integral), a field of symmetries ($\mathbf{w} = \rho^{-1}(\text{rot}\,\mathbf{v})$, where ρ is the density of the fluid) and an integral invariant (the circulation of the fluid around a closed fluid contour). The field of symmetries was determined in [5]; in the case of a uniform fluid ($\rho \equiv$ const), it was pointed out before that in [6].

A necessary condition for the Bernoulli integral and the aforementioned field of symmetries to be non-trivial is that the field **v** should not be collinear with its curl. This condition is essential. For example, it is not satisfied for the well-known ABC-flow, and this flow is generally not regular.

It has been shown [7] that typical steady flows of a viscous fluid are not regular and possess chaotic properties. On the other hand, as we know, turbulence occurs at high Reynolds numbers. At fixed characteristic scales and flow velocity, this is equivalent to reducing the viscosity. However, as already indicated, at zero viscosity, typical steady fluid flows become regular.

2. CONDITIONS FOR THE EXISTENCE OF TENSOR INVARIANTS

In a small neighbourhood of any non-singular point, system (1.1) has all tensor invariants: a non-constant integral, a non-trivial field of symmetries and an integral invariant. The situation changes radically when one considers the problem of whether invariants exist in the large (i.e. are defined in the entire flow domain D). It turns out that dynamical systems of general form do not have non-trivial global invariants. This fact is of paramount importance in continuum mechanics.

Example. Suppose the fluid is incompressible. It then follows from the equation of continuity that the density ρ is a first integral. If Eqs (1.1) do not admit of non-constant integrals, then $\rho \equiv$ const (that is, the fluid is uniform). In the case of regular flows, generally speaking, $\rho \not\equiv$ const.

The derivation of constructive conditions for the regularity or chaoticity of a dynamical system (1.1) is a very complicated problem. It has been discussed for the dynamical systems of classical dynamics in [8].

Let us confine our attention in the domain $\mathbb{T}^2 \times \mathbb{R}$ to autonomous systems of the following form:

$$d\mathbf{x}/dt = \mathbf{v}_0 + \varepsilon\mathbf{v}_1 + \ldots; \quad \mathbf{x} = (x, y, z), \quad \mathbf{v}_i = (u_i, v_i, w_i), \quad i = 0, 1, \ldots, \quad w_0 \equiv 0 \tag{2.1}$$

where u_i, v_i and w_i are analytic functions where are 2π-periodic in x and y; u_0 and v_0 depend only on the "slow" variable z and ε is a small parameter. Systems of this form are frequently encountered in the theory of linear oscillations. When $\varepsilon = 0$ we have a completely integrable (and therefore regular) dynamical system: its phase space is stratified into invariant tori z = const filled by conditionally periodic trajectories with frequencies $u_0(z)$ and $v_0(z)$.

The tensor invariants of system (2.1) are naturally sought as series in powers of ε with coefficients which are single-valued and analytic in the domain $\mathbb{T}^2 \times \mathbb{R}$. For example, the integral has the form $f_0 + \varepsilon f_1 + \ldots$, where f_k are analytic 2π-periodic functions and x and y.

Let $\Sigma W_{mn}(z)e^{i(mx+ny)}$ be the Fourier series of the function w_1. The *Poincaré set* $\mathbb{P}$ is defined as the set of points $z \in \mathbb{R}$ satisfying the conditions

$$mu_0(z) + nv_0(z) = 0, \quad W_{mn}(z) \neq 0 \quad (m^2 + n^2 \neq 0) \tag{2.2}$$

In the general case, the set $\mathbb{P}$ is everywhere dense in the real line $\mathbb{R} = \{z\}$. Relations of type (2.2) with integers m and n are called *resonances*. The unperturbed system (when $\varepsilon = 0$) is said to be *non-degenerate* if $mu_0(z) + nv_0(z) \not\equiv 0$ for all integers m and n not both zero.

A simple sufficient condition for non-degeneracy is the following: $u_0'v_0 - u_0v_0' \not\equiv -0$. The prime denotes differentiation with respect to z.

Theorem 1. Let us assume that the unperturbed system is non-degenerate and that the Poincaré set has at least one finite limit point. Then system (2.1) does not admit of non-trivial integrals and frozen-in fields of directions analytic in ε. If moreover $W_{00}(z) \neq 0$, then system (2.1) does not admit of non-trivial integral invariants.

The absence of integrals follows from the more general results of [8], which generalize Poincaré's theory of obstructions to the integrability of Hamiltonian systems that differ slightly from completely integrable systems [9]. Cases have been established in which no frozen-in fields of directions and integral invariants exist [3, 7]. The following point is worthy of note: the conditions of Theorem 1 guarantee the absence of non-trivial invariants which can be expressed as *formal* (not necessarily convergent) power series in ε.

The chaotization mechanism of system (2.1) consists of the fact that typical resonant tori are destroyed when a perturbation is applied: these tori are replaced by "islets" with chaotic behaviour of the trajectories, and their dimensions for small ε values increase as a rule as ε increases; in addition, the typical size of the "islets" of instability decreases as the order of the resonance $|m| + |n|$ increases. Under certain additional conditions, the resonant tori are replaced by pairs of *non-degenerate* periodic trajectories, one elliptic and the other, hyperbolic. The intersecting separatrices of the hyperbolic periodic trajectories form a tangled net, in whose neighbourhood there are quasi-random trajectories (for examples from hydrodynamics see [7]).

There is a simpler sufficient condition for a non-degenerate system (2.1) to have no first integrals which are analytic functions of the parameter ε

$$\langle w_1 \rangle = W_{00}(z) \not\equiv 0 \tag{2.3}$$

Indeed, let $f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots$ be a non-constant first integral. In the light of our assumption that system (2.1) is non-degenerate when $\varepsilon = 0$, the function f_0 is independent of the angular coordinates x and y [8]. In the first approximation in ε, the condition $df/dt = 0$ becomes

$$f_0' w_1 + u_0 \partial f_1 / \partial x + v_0 \partial f_1 / \partial y = 0$$

After averaging over the angular coordinates x and y, we obtain the equation $f_0' W_{00} = 0$. Since there are no divisors of zero in the ring of analytic functions, it follows from condition (2.3) that $f_0' = 0$. Consequently, $f_0 \equiv \text{const}$ and function $(f - f_0)/\varepsilon = f_1 + \varepsilon f_2 + \ldots$ will be an integral of system (2.1). We again obtain $f_1 \equiv \text{const}$. Continuing the process, we obtain the equalities $f_n \equiv \text{const}$, $n \geqslant 0$. Consequently, $f \equiv \text{const}$.

Note that condition (2.3) is formal in nature and unconnected with the chaotization of the trajectories of system (2.1). If system (2.1) was obtained from a Hamiltonian system after reducing the Whittaker order, then condition (2.3) is surely not satisfied [8].

Here is a simple example of a system satisfying condition (2.3) which has integrals that are non-analytic functions of the parameter ε

$$dx/dt = 1, \quad dy/dt = v_0(z) + \varepsilon v_1 + \ldots, \quad dz/dt = \varepsilon \tag{2.4}$$

Here $\langle w_1 \rangle = 1$, and therefore the system does not admit of single-valued first integrals analytic in ε. However, for all $\varepsilon \neq 0$ it has an integral $\sin(x - z/\varepsilon)$ which is a 2π-periodic function of x and is not analytic in ε when $\varepsilon = 0$.

Note that system (2.4) admits of an analytic invariant 1-form $\omega = \varepsilon dx - dz$. This does not contradict Theorem 1, since here the Poincaré set is empty.

The presence of finite limit points of the Poincaré set apparently implies that no non-constant analytic invariants exist at small fixed values of $\varepsilon \neq 0$. However, this remains to be proved.

3. APPLICATION TO BURGERS EQUATIONS

A general method has been proposed to investigate the regularity property of typical steady flows of an incompressible viscous fluid [7]. One looks for solutions of the Navier–Stokes equations as series in powers of the parameter ε of the form of (2.1), and then applies Theorem 1. It turns out that typical steady flows are chaotic (in the sense indicated in Section 2), though some flows (such as Hagen–Poiseuille flow in a cylinder) may be regular. This result shows that the classical results of Bernoulli, Helmholtz, and Thomson on invariants of flows of an ideal fluid cannot be extended to the case of a viscous fluid.

It has been shown [7] that this conclusion also holds for the simplified Navier–Stokes equations ignoring inertial terms (the Stokes approximation).

Let us demonstrate the possibilities of applying the method to the *Burgers equations*, which are also a simplification of the Navier–Stokes system (a fluid without pressure). These equations are [10]

$$d\mathbf{v}/dt = \nu\Delta\mathbf{v} \tag{3.1}$$

where $\nu = \text{const} > 0$ is the kinematic coefficient of viscosity. These equations have a series solution

$$\mathbf{v} = \mathbf{v}_0 + \varepsilon\mathbf{v}_1 + \ldots, \quad \mathbf{v}_0 = (u_0, v_0, 0), \quad \mathbf{v}_1 = (u_1, v_1, w_1)$$

where

$$u_0 = \alpha z + \beta, \quad v_0 = \gamma z + \delta \tag{3.2}$$

and α, β, γ and δ are arbitrary constants. The function $\mathbf{v}_1$ satisfies the linear differential equation

$$L\mathbf{v}_1 + \boldsymbol{\alpha} w_1 = \nu\Delta\mathbf{v}_1; \quad L = u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}, \quad \boldsymbol{\alpha} = (\alpha, \gamma, 0)$$

This equation is easily solved by the Fourier method. Let U_{mn}, V_{mn} and W_{mn} be the Fourier coefficients of the velocity components u_1, v_1 and w_1, respectively; they are functions of z. These coefficients satisfy a linear system of ordinary differential equations, which we write in vector form

$$i(mu_0 + nv_0)\mathbf{V}_{mn} + \boldsymbol{\alpha} W_{mn} = \nu[-(m^2 + n^2)\mathbf{V}_{mn} + \mathbf{V}''_{mn}], \quad \mathbf{V}_{mn} = (U_{mn}, V_{mn}, W_{mn}). \tag{3.3}$$

Each solution of this system is defined on the entire real line $\mathbb{R} = \{z\}$ and is uniquely defined by the values of U_{mn}, V_{mn}, W_{mn} and their derivatives at some point Z_{mn}. Set

$$z_{mn} = -(m\beta + n\delta)/(m\alpha + n\gamma) \tag{3.4}$$

This point is a root of the equation $mu_0 + nv_0 = 0$.

Suppose that

$$\alpha\delta - \gamma\beta \neq 0 \tag{3.5}$$

This is the condition for the unperturbed system to be non-degenerate. In addition, taking condition (3.5) into consideration, we conclude that the set of points (3.4) is everywhere dense in the real line $\mathbb{R} = \{z\}$. If $W_{mn}(z_{mn}) \neq 0$ (the typical case), the Poincaré set is everywhere dense and therefore (by Theorem 1) a typical steady flow in the Burgers model has no non-constant integrals and non-trivial frozen-in-fields of directions. Furthermore, when $m = n = 0$ the system of equations (3.3) is non-degenerate and we may therefore assume in the general case that $W_{00} \neq 0$. In the general case, therefore, there are no non-trivial integral invariants either.

Now put $\nu = 0$. Then the Burgers equations (3.1) become the Hopf equations, which describe the inertial motion of a collisionless continuous medium:

$$\partial\mathbf{v}/\partial t + \mathbf{v}\partial\mathbf{v}/\partial\mathbf{x} = \mathbf{0}$$

By condition (3.5), one of the numbers α or γ must not vanish. Consequently, it follows from (3.3) that $W_{mn}(z) = 0$ for $z = z_{mn}$. In the case under consideration, therefore, the Poincaré set is empty and Theorem 1 is not applicable.

Of course, this is no accident: For the Hopf system, Eqs (1.1) admit of an integral $f = (\mathbf{v}, \mathbf{v})$, a field of symmetries $\mathbf{w} = \text{rot}\mathbf{v}$, and an integral invariant

$$\int (\mathbf{v}, d\mathbf{x}) = \text{const}$$

4. THE NECESSARY CONDITION FOR CHAOTICITY OF FLOWS OF AN IDEAL FLUID

It is instructive to apply the method developed above to Euler's equations for the flow of an ideal fluid. In that case Theorem 1 yields a condition for the chaotization of steady flows: the velocity field is collinear with its curl.

To simplify the notation, we will confine ourselves to the case of a uniform fluid ($\rho \equiv$ const). The equations of motion in an irrotational field have the form

$$d\mathbf{v}/dt = -\partial f/\partial \mathbf{x}, \qquad \operatorname{div} \mathbf{v} = 0 \tag{4.1}$$

where $f = p/\rho + V$, p is the pressure and V is the potential of external bulk forces.

We will seek a solution of Eqs (4.1) in series form

$$\mathbf{v} = \mathbf{v}_0 + \varepsilon \mathbf{v}_1 + \ldots, \quad w_0 = 0, \quad f = f_0 + \varepsilon f_1 + \ldots, \quad f_0 = \text{const}$$

where u_0 and v_0 are arbitrary analytic functions of z.

In the first approximation in ε, we have the following system of partial differential equations (the prime denotes differentiation with respect to z)

$$L\mathbf{v}_1 + \mathbf{v}_0' w_1 = -\partial f_1 / \partial \mathbf{x}, \quad \operatorname{div} \mathbf{v}_1 = 0$$

Solving this equation by Fourier's method, we obtain the equations

$$\begin{aligned}
&i(mu_0 + nv_0)U_{mn} + u_0' W_{mn} + imF_{mn} = 0 \\
&i(mu_0 + nv_0)V_{mn} + v_0' W_{mn} + inF_{mn} = 0 \\
&i(mu_0 + nv_0)W_{mn} + F_{mn}' = 0 \\
&i(mU_{mn} + nV_{mn}) + W_{mn}' = 0
\end{aligned} \tag{4.2}$$

Let us assume that the assumptions of Theorem 1 are satisfied and that steady flow does not admit of invariants as power series in ε. Then the Poincaré set $\mathbb{P}$ contains infinitely many distinct points z_{mn} that accumulate in a finite interval of the real axis $\mathbb{R} = \{z\}$. Since at these points $mu_0 + nv_0 = 0$, but $W_{mn} \neq 0$, it follows from the first two equations of system (4.2) that $nu_0' - mv_0' = 0$. Hence the following equation is true on $\mathbb{P}$

$$u_0 u_0' + v_0 v_0' = 0 \tag{4.3}$$

Since the set $\mathbb{P}$ has finite limit points, and the function u_0 and v_0 are assumed to be analytic, Eq. (4.3) is true everywhere. It remains to observe that Eq. (4.3) is equivalent to the condition that the vector fields $\operatorname{rot}\mathbf{v}$ and $\mathbf{v}$ are collinear when $\varepsilon = 0$.

In fact, the collinearity of these fields in the first approximation in ε may be derived from system (4.2). For example, the collinearity condition projected onto the z axis is

$$\partial v_1 / \partial x - \partial u_1 / \partial y = \lambda_0 w_1, \quad \lambda_0 = \lambda_0(z)$$

This is equivalent to an infinite chain of algebraic relations for the Fourier coefficients

$$i(mV_{mn} - nU_{mn}) = \lambda_0 W_{mn} \tag{4.4}$$

On the other hand, it follows from Eq. (4.3) that $u_0' = \mu v_0$, $v_0' = -\mu u_0$, where μ is some function. Substituting these relations into Eqs (4.2) and using the fact that the unperturbed system is non-degenerate and that the ring of analytic functions contains no divisors of zero, we obtain (4.4), in which $\lambda_0 = \mu$.

5. STEADY FLOWS OF A PERFECT GAS

Let us apply our method to the case of a perfect gas, whose dynamics is described by the following closed system of equations

$$\begin{aligned}
&\rho d\mathbf{v}/dt = -\partial p/\partial \mathbf{x} + \rho \mathbf{F}, \quad d\rho/dt + \rho \operatorname{div} \mathbf{v} = 0, \quad p = b\rho\tau \\
&\rho a d\tau/dt = b\tau d\rho/dt + \varkappa \Delta \tau
\end{aligned} \tag{5.1}$$

The first equation is Euler's equation ($\mathbf{F}$ is the external force), the second is the equation of continuity, the third is the equation of state (τ is the absolute temperature and b is the gas constant), and the fourth is the heat flux equation (a and $\varkappa$ = const are the heat capacity and thermal conductivity, respectively).

We will first consider the simple case when $\varkappa = 0$. The fourth equation of (5.1) then immediately yields the integral

$$g = \tau / \rho^{\lambda}, \quad \lambda = b / a \tag{5.2}$$

If the external forces are irrotational ($\mathbf{F} = -\partial V/\partial \mathbf{x}$), one has a generalized Bernoulli integral

$$f = (\mathbf{v}, \mathbf{v})/2 + (a+b)\tau + V$$

This means that when $\varkappa = 0$ the typical steady flows of a perfect gas are regular. Moreover, if the flow domain D is compact and the functions f and g are almost everywhere independent, then the trajectories of almost all fluid particles are closed: the particles move in closed orbits, generally with different periods.

The situation changes radically when $\varkappa \neq 0$; a typical steady gas flow in an arbitrary rotational force field does not generally admit of non-constant integrals.

Let X, Y and Z be the components of the vector field $\mathbf{F}$; they are functions of x, y and z which are 2π-periodic in x and y. Let us consider the case in which they are represented in the form of series in powers of the parameter ε

$$X = \varepsilon X_1 + \ldots, \quad Y = \varepsilon Y_1 + \ldots, \quad Z = Z_0 + \varepsilon Z_1 + \ldots$$

Where Z_0 is an analytic function of z. When $\varepsilon = 0$ this force field is irrotational.

We will seek solutions of system (5.1) a power series

$$\begin{aligned} &\mathbf{v} = \mathbf{v}_0 + \varepsilon \mathbf{v}_1 + \ldots, \quad \rho = \rho_0 + \varepsilon \rho_1 + \ldots, \quad \tau = \tau_0 + \varepsilon \tau_1 + \ldots \\ &\mathbf{v}_0 = (u_0, v_0, 0), \quad \mathbf{v}_1 = (u_1, v_1, w_1) \end{aligned} \tag{5.3}$$

The function τ_0 depends linearly on z; u_0 and v_0 are arbitrary functions of z and ρ_0 is found as a function of the height z from the equation

$$Z_0 = b(\rho_0 \tau_0)' / \rho_0$$

We will confine ourselves to the special case in which u_0 and v_0 are linear functions of z of the form (3.2), with $\alpha\delta - \gamma\beta \neq 0$. We again define the numbers z_{mn} by formula (3.4).

Let X_{mn} and Y_{mn} be the Fourier coefficients of the functions X_1 and Y_1, respectively.

Theorem 2. If $nX_{mn} - mY_{mn} \neq 0$ ($m^2 + n^2 \neq 0$) for $z = z_{mn}$, then the aforementioned steady flows of a perfect gas do not admit of non-trivial invariants analytic in ε.

It is clear that rotational force fields of general form satisfy the assumption of Theorem 2. The proof relies on Theorem 1.

Substituting expansions (5.3) into system (5.1), we obtain a chain of partial differential equations of the successive determination of the coefficients. In the first approximation in ε, we obtain a linear system in $u_1 \ldots \rho_1, \tau_1$, which is easily solved by Fourier's method. The Fourier coefficients $U_{mn}, \ldots \mathcal{P}_{mn}, \mathcal{T}_{mn}$ satisfy the following system algebraic-differential equations

$$\begin{aligned} &i(mu_0 + nv_0)U_{mn} + u_0' W_{mn} = -ibm\Delta_{mn} + X_{mn} \\ &i(mu_0 + nv_0)V_{mn} + v_0' W_{mn} = -ibn\Delta_{mn} + Y_{mn} \\ &i(mu_0 + nv_0)W_{mn} = -b\Delta_{mn}' - b\rho_0'\rho_0^{-1}\mathcal{T}_{mn} + b\tau_0'\rho_0^{-1}\mathcal{P}_{mn} + Z_{mn} \\ &i(mu_0 + nv_0)\mathcal{P}_{mn} + \rho_0' W_{mn} + \rho_0(imU_{mn} + inV_{mn} + W_{mn}') = 0 \\ &a[i(mu_0 + nv_0)\mathcal{T}_{mn} + \tau_0' W_{mn}] = b\tau_0\rho_0^{-1}[i(mu_0 + nv_0)\mathcal{P}_{mn} + \rho_0' W_{mn}] + \\ &+\varkappa\rho_0^{-1}[\mathcal{T}_{mn}'' - (m^2 + n^2)\mathcal{T}_{mn}] \end{aligned} \tag{5.4}$$

where $\Delta_{mn} = \mathcal{T}_{mn} + \rho_0^{-1}(\tau_0 \mathcal{P}_{mn})$.

The first two equations uniquely define the coefficients U_{mn} and V_{mn}. These functions will be analytic on the line $\mathbb{R} = \{z\}$, if and only if the following equations hold when $z = z_{mn}$

$$\alpha W_{mn} + m(ib\Delta_{mn}) = X_{mn}, \quad \gamma W_{mn} + n(ib\Delta_{mn}) = Y_{mn}$$

By the non-degeneracy condition, the determinant of this system does not vanish when $m^2 + n^2 \neq 0$. Consequently, the quantities W_{mn} and Δ_{mn} are uniquely defined at z_{mn}, and (by the assumption of Theorem 2) $W_{mn}(z_{mn}) \neq 0$.

Substituting the expressions found for U_{mn} and V_{mn} into the other equations of (5.4), we obtain a system of linear differential equations for W_{mn}, $\mathcal{P}_{mn}$ and $\mathcal{T}_{mn}$, of orders 1, 1, and 2, respectively. Its solutions, which are analytic on the whole line $\mathbb{R} = \{z\}$, are uniquely defined, e.g. by specifying the values of $\mathcal{T}_{mn}$ and $\mathcal{T}'_{mn}$ at the points $z = z_{mn}$.

Since the Poincaré set $\mathbb{P} = \{z_{mn}\}$ is everywhere dense in $\mathbb{R}$, it remains to apply Theorem 1.

It is interesting to consider the case when $\varkappa = 0$ from this point of view. It follows from the last equation of (5.4) that the following equation holds when $z = z_{mn}$

$$[a\tau_0' - b\tau_0\rho_0^{-1}\rho_0']W_{mn} = 0$$

The zeros of the bracketed expression are the critical points of the function (5.2). In the general case, this function is not constant. Since the critical points of an analytic function cannot accumulate in a finite domain, Theorem 1 is obviously not applicable in this case.

6. APPENDIX. SOME INTEGRAL RELATIONS

Let us assume that the domain D is compact and its boundary ∂D is a smooth regular surface. Let $\mathbf{n}$ be the inner unit normal vector to the boundary surface.

If a heat conducting medium is stationary ($\mathbf{v} = 0$) and $\partial\tau/\partial\mathbf{n} = 0$ on the boundary ∂D, then $\tau = \text{const}$ throughout the domain D. Indeed, by the heat flux equation, the temperature in that case will be a harmonic function in the domain D. It remains to use a well-known property of solutions of the interior Neumann problem (see, e.g. [11]).

It turns out that this result also holds for steady flows of a perfect gas. The proof uses a technique that may also prove useful in other problems of this kind. Because of the impermeability property, the velocity field $\mathbf{v}$ is tangent to the boundary ∂D.

Lemma. Let $f: D \to \mathbb{R}$ be a continuous function. Then

$$\int_D \rho\left(\frac{\partial f}{\partial \mathbf{x}}, \mathbf{v}\right) d^3\mathbf{x} = 0 \tag{6.1}$$

Indeed

$$(\partial f / \partial \mathbf{x}, \rho\mathbf{v}) = \operatorname{div}(f\rho\mathbf{v}) - f \operatorname{div}(\rho\mathbf{v})$$

By the equation of continuity, $\operatorname{div}(\rho\mathbf{v}) = 0$. Formulae (6.1) now follows from Gauss' theorem.

Another proof of the theorem is based on the use of the Ergodic Theorem. Since D is compact, the function f is bounded. Consequently, the time average of the function $df/dt = (f'_x, \mathbf{v})$ is zero. The dynamical system (1.1) has an invariant measure $\rho d^3\mathbf{x}$. It remains to apply the individual Birkhoff–Khinchin Ergodic Theorem (see, e.g. [12]) to the function df/dt.

The heat flux equation (the fourth equation in system (5.1)) may be written as follows:

$$d(a \ln \tau - b \ln \rho) / dt = \varkappa\Delta\tau(\rho\tau)^{-1}$$

By the lemma

$$\int_D \frac{\Delta\tau}{\tau} d^3\mathbf{x} = 0 \tag{6.2}$$

Theorem 3. If

$$\int_{\partial D} \frac{\partial \ln \tau}{\partial \mathbf{n}} d\sigma = 0 \tag{6.3}$$

then $\tau = \text{const}$ in the domain D.

Corollary. If $\partial\tau/\partial\mathbf{n} = 0$, then $\tau = \text{const}$.

Proof. Applying Green's preparation formula to the integral on the left hand side of (6.2), we obtain

$$\int_{\partial D} \frac{1}{\tau}\frac{\partial\tau}{\partial\mathbf{n}}d\sigma = \int_D \frac{1}{\tau^2}\left(\frac{\partial\tau}{\partial\mathbf{x}}\right)^2 d^3\mathbf{x} \tag{6.4}$$

By condition (6.3), the left-hand side of this equality vanishes. Since the integrand on the right of (6.4) is non-negative and the integral vanishes, it follows that $\partial\tau/\partial\mathbf{x} = 0$. Hence it follows that $\tau = \text{const}$.

Let us consider the special case in which the gas flow is periodic in all the coordinates x, y and z. Such a flow may be considered as flow in a three-dimensional "flat" torus $\mathbb{T}^3$. Since $\partial\mathbb{T}^3$ is the empty set, condition (6.3) is obviously satisfied. Consequently, by theorem 3, $\tau = \text{const}$. It follows from the last three equations of system (5.1) that the gas is a barotropic incompressible fluid. It was shown in [5] that if the velocity field is not collinear with its curl, such a steady flow in an irrotational force field will be regular. Note that the gas flows of Theorem 2 are assumed to be periodic in only two of the coordinates, x and y.

This research was supported financial by the Russian Foundation for Basic Research (02-01-01059) and the "State of Leading Scientific Schools" Program (00-15-96146).

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Translated by D.L.