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EVOLUTION OF MEASURES IN THE PHASE SPACE OF NONLINEAR HAMILTONIAN SYSTEMS

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We establish the existence of weak limits of solutions (in the class L_p, $p \geq 1$) of the Liouville equation for nondegenerate quasihomogeneous Hamilton equations. We find the limit probability distributions in the configuration space. We give conditions for a uniform distribution of Gibbs ensembles for geodesic flows on compact manifolds.

Keywords: quasihomogeneous Hamiltonian system, geodesic flow, weak limit, Gibbs ensemble, uniform distribution

1. Introduction

Let Γ be the phase space of a dynamical system

$$\frac{dz}{dt} = v(z), \quad z \in \Gamma, \tag{1}$$

and g^t be its phase flow preserving a measure μ. In what follows, we always assume that the vector fields under consideration determine dynamical systems in the sense that the corresponding phase flows g^t are defined for all $t \in \mathbb{R}$. Let

$$1 \leq p, q \leq \infty, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{2}$$

We consider two functions $f_p, f_q \colon \Gamma \to \mathbb{R}$, $f_p \in L_p(\Gamma, \mu)$, $f_q \in L_q(\Gamma, \mu)$. Because the spaces L_p and L_q are conjugate to each other, there is the operation

$$(f_p, f_q) = \int_\Gamma f_p(z) f_q(z)\, d\mu(z).$$

Because g^{-t} preserves the measure μ, $f_p \circ g^{-t} \in L_p(\Gamma, \mu)$ for all t.

We consider the function of time given by

$$k(t) = (f_p \circ g^{-t}, f_q). \tag{3}$$

An important role in what follows is played by the limit case where $p = 1$, $q = \infty$ (we recall that the space L_∞ consists of measurable essentially bounded functions).

In what follows, we study the conditions under which $k(t)$ has a limit as $t \to \pm\infty$. Let $\mu(\Gamma) < \infty$ and (3) be a system with intermixing. Then

$$\lim_{t \to \pm\infty} k(t) = \frac{1}{\mu(\Gamma)} \int_\Gamma f_p\, d\mu \int_\Gamma f_q\, d\mu. \tag{4}$$

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Obviously, by the finiteness assumption for the measure $\mu(\Gamma)$, the functions f_p and f_q are integrable.

It is well known that equality (4) holds for $p = q = 2$. For arbitrary p and q satisfying (2), this can be easily proved using the following argument. For definiteness, we assume that $p < 2 < q$. For any $\varepsilon > 0$, we represent the function f_p as $f_p = \hat{f}_p + \tilde{f}_p$, where $\hat{f}_p \in L_2(\Gamma, \mu)$ and $\|\tilde{f}_p\|_{L_p} < \varepsilon$ (as $\hat{f}_p$, we can take, e.g., a cutoff function of f_p). Then

$$k(t) = (\hat{f}_p \circ g^{-t}, f_q) + r, \qquad r = (\tilde{f}_p \circ g^{-t}, f_q).$$

It remains to note that $f_q \in L_2(\Gamma, \mu)$ and $|r| \leq \varepsilon \|f_q\|_{L_q}$. This limit may not exist for systems without intermixing (even ergodic ones).

Example 1. Let Γ be the n-dimensional torus $\mathbb{T}^n = \{z_1, \dots, z_n \pmod{2\pi}\}$ and system (1) be given by the equations

$$\dot{z}_1 = \omega_1, \qquad \dots, \qquad \dot{z}_n = \omega_n$$

with constant frequencies $\omega_1, \dots, \omega_n$ that are independent over the ring of integers. If f_p and f_q are characteristic functions of measurable domains on $\mathbb{T}^n$, then $k(t)$ oscillates and has no limit as $t \to \pm\infty$.

A generalization of the previous example is as follows.

Example 2. Let $v(x)$ be an arbitrary vector field on a smooth manifold M preserving a measure μ. We consider the dynamical system

$$\dot{x} = v(x), \qquad \dot{\varphi} = 1$$

on $M \times \mathbb{T}$, where $\mathbb{T} = \{\varphi \pmod{2\pi}\}$ is the one-dimensional torus. If the supports of f_p and f_q are mapped under the natural projection $M \times \mathbb{T} \to \mathbb{T}$ into the interior of some segments

$$I_p, I_q \subset \mathbb{T}, \quad \mathbb{T} \setminus I_p \neq \varnothing, \quad \mathbb{T} \setminus I_q \neq \varnothing,$$

then $k(t)$, generally speaking, oscillates and has no limit as $t \to \pm\infty$.

Example 3. Let $T\colon M \to M$ be a diffeomorphism preserving a measure μ on a smooth compact manifold M. We recall the standard construction of a suspension over T. Let $l\colon M \to (0, \infty)$ be a smooth function. In the direct product $M \times \mathbb{R}$, we consider the subset

$$\widetilde{M}_l = \{(x, s) \in M \times \mathbb{R}\colon s \in [0, l(x)]\}.$$

As a result of the identification $(x, l(x)) \sim (T(x), 0)$, this subset becomes a smooth manifold M_l, on which the vector field $\partial/\partial s$ is defined. The obtained system is called the suspension over T. The corresponding phase flow preserves the measure $d\mu\, ds$ on M_l, where ds is the Lebesgue measure with respect to the coordinate s. If $l = \text{const}$, the arguments adduced in Example 1 remain valid. In particular, the function $k(t)$ typically has no limit, as before.

Proposition. *Let the limit* $\lim_{t\to\infty} k(t) = k_\infty$ *exist for some functions* $f_p \in L_p(\Gamma, \mu)$ *and* $f_q \in L_q(\Gamma, \mu)$. *Then*

$$k_\infty = (\bar{f}_p, f_q), \tag{5}$$

where

$$\bar{f}_p(z) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f_p \circ g^{-t}(z)\, dt. \tag{6}$$

The existence of limit (6) for almost all z in the case where $p = 1$ follows from the Birkhoff ergodic theorem. The generalization to the case where $p \neq 1$ can be found in [1], [2]. The function $\bar{f}_p$ is invariant under g^t. If $\mu(\Gamma) < \infty$ and $p \geq 1$, then

$$\int_\Gamma f\, d\mu = \int_\Gamma \bar{f}\, d\mu.$$

For $p = q = 2$, the above proposition is established in [3] (the proof is contained in [4]). In the general case, the proof is similar. For completeness, we give it in the appendix.

If system (1) is ergodic, then

$$\bar{f}_p = \frac{1}{\mu(\Gamma)} \int_\Gamma f_p\, d\mu,$$

and (4) is implied by (6).

We consider systems (1) of the form

$$\frac{dz}{dt} = v(z, \omega), \qquad \frac{d\omega}{dt} = 0. \tag{7}$$

The phase space Γ is the direct product $\Lambda \times \Delta$, where Λ is a smooth n-dimensional manifold and Δ is an interval (possibly, infinite) on the real axis $\mathbb{R}$. The coordinate $\omega \in \Delta$ is a first integral. We assume that for a fixed ω, the system has an invariant measure $d\nu = \lambda(z, \omega)\, d^n z$ on Λ.

In particular, this applies to Hamiltonian systems: the role of the coordinate ω is played by the total energy and Λ is a nonsingular energy surface. In this case, the phase space Γ of the Hamiltonian system with the Hamiltonian H is divided into cells $h_1 \leq H \leq h_2$ such that there are no critical values of H in the interval (h_1, h_2).

We recall the definition of a dynamical system with a stratified flow [4]. We set $P_\gamma = \{(z, \omega) \in \Gamma \colon \omega = \gamma\}$. These are n-dimensional integral manifolds of system (7). The mapping $\psi_\omega \colon (z, \omega) \mapsto z$ determines a natural diffeomorphism that takes P_ω to Λ. The vector field v is tangent to P_ω. We let v_ω denote the restriction of v to P_ω and let g^t_ω be its phase flow.

Definition. A flow g^t is said to be *stratified* if there exists a smooth function $\alpha \colon \Delta \to (0, \infty)$ and a flow $g^s_* \colon \Lambda \to \Lambda$ such that the diagram

$$\begin{array}{ccc} P_\omega & \xrightarrow{g^t_\omega} & P_\omega \\ {\scriptstyle\psi_\omega}\big\downarrow & & \big\downarrow{\scriptstyle\psi_\omega} \\ \Lambda & \xrightarrow{g_*^{\alpha(\omega)t}} & \Lambda \end{array} \tag{8}$$

is commutative for all $\omega \in \Delta$ and all $t \in \mathbb{R}$. A stratified flow is said to be *nondegenerate* if the function $\alpha(\omega)$ has only isolated critical points.

Identifying P_ω and Λ via the diffeomorphism ψ_ω, we can express the commutativity of diagram (8) as

$$g^t_\omega = g_*^{\alpha(\omega)t}. \tag{9}$$

Examples of systems with stratified flows are given in [4]. In particular, geodesic flows on Riemannian manifolds (describing the inertial motion of natural mechanical systems) are such systems.

Example 4. A Hamiltonian system

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad i = 1, \dots, m, \tag{10}$$

is said to be *quasihomogeneous* if it is invariant under the similarity transformation

$$t \mapsto s^{-1}t, \qquad q \mapsto s^a q, \qquad p \mapsto s^b p, \qquad H \mapsto s^c H,$$

where $s > 0$ is a parameter. The quasihomogeneity weights a, b, and c satisfy the natural relation $a+b+1 = c$. The manifolds P_ω are defined as the point sets $\{p, q \colon H(p,q) = \omega\}$.

For all $\omega > 0$ ($\omega < 0$), these manifolds are diffeomorphic. For example, let $\omega > 0$. We set $\omega = s^c$, $s > 0$. As the manifold Λ, we can then take the energy surface $\{H = 1\}$; the inverse diffeomorphism ψ_ω^{-1} is given by

$$q \mapsto s^a q, \qquad p \mapsto s^b p,$$

with $\alpha(\omega) = 1/s = \omega^{-1/c}$. Therefore, if $c \neq 0$, then the flow of quasihomogeneous Hamiltonian system (10) is nondegenerate.

In particular, the phase flow of the problem of n gravitating bodies is a nondegenerate stratified flow. Here, $a = -2/3$, $b = 1/3$, and $c = 2/3$. In the problem of inertial motion, the Hamilton function is a homogeneous quadratic form in the momenta. The corresponding Hamilton equations are also quasihomogeneous with the weights $a = 0$, $b = 1$, and $c = 2$.

If the flow g_*^t in (8) preserves the measure ν_* on the manifold Λ and if σ is any measure on the interval Δ, then the flow g^t on $\Lambda \times \Delta$ preserves the measure $\mu = \nu_* \times \sigma$. The following theorem is our main result.

Theorem 1. *Let g^t be a nondegenerate stratified flow on $\Gamma = \Lambda \times \Delta$, the measure ν_* be absolutely continuous with respect to the measure on Λ determined by a Riemannian metric, and the measure σ be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Here, $\nu_*(\Lambda) < \infty$. Then for any $f_p \in L_p(\Gamma, \mu)$ and $f_q \in L_q(\Gamma, \mu)$ with p and q satisfying conditions* (2), *there exists*

$$\lim_{t \to \pm\infty} k(t) = (\bar{f}_p, f_q).$$

For $p = q = 2$, Theorem 1 is proved in [4]. In the general case, the proof is similar (see below) but can hardly be obtained by simply reducing to the case where $p = q = 2$, for example, by approximating f_p and f_q by functions in $L_2(\Gamma, \mu)$. In any event, it seems impossible to avoid referring to highly nontrivial results in [1], [2], obtained without direct reduction to the case of standard ergodic theorems. Theorem 1, in particular, implies that for systems with nondegenerate stratified flows, the function $f_p \circ g^t$ *weakly converges* to its Birkhoff average $\bar{f}_p$ as $t \to \pm\infty$. For quasihomogeneous Hamiltonian systems, this implies that the mechanical system irreversibly tends to the statistical (i.e., thermal) equilibrium (according to Gibbs). We note that we assumed that $\sigma(\Delta) < \infty$ in [4]. It later became clear that requiring that the measure be finite is irrelevant.

2. New form of the ergodic theorem

We again consider dynamical system (1) with an invariant measure μ. Let $h(\omega)$ be the density of a probabilistic measure on $\mathbb{R} = \{\omega\}$: this is a nonnegative function of $L_1(\mathbb{R}, d\omega)$ such that

$$\int_{-\infty}^{\infty} h(\omega)\, d\omega = 1.$$

Theorem 2. *If $f_p \in L_p(\Gamma, \mu)$ and $f_q \in L_q(\Gamma, \mu)$, then*

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} h(\omega)(f_p \circ g^{\omega t}, f_q)\, d\omega = (\bar{f}_p, f_q).$$

Theorem 2 is a special case of Theorem 1. Indeed, we replace the original Eq. (1) with a system of form (7),

$$\dot{z} = \omega v(z), \qquad \dot{\omega} = 0. \tag{11}$$

The corresponding phase flow is stratified with $\alpha(\omega) = \omega$. Having made this remark, we can replace f_p and f_q in Theorem 1 with $h^{1/p} f_1$ and $h^{1/q} f_2$. We first prove Theorem 2 (see Sec. 5) and then derive Theorem 1 from it (see Sec. 6).

3. Distribution density in the configuration space

Let M be a compact configuration space (possibly, with a boundary) of a mechanical system with n degrees of freedom, and let $x = (x_1, \ldots, x_n)$ be local coordinates on it. The phase space Γ is the bundle cotangent to M ($\Gamma = T^*M$). We consider inertial motion, and the Hamilton function therefore amounts to the kinetic energy term

$$H = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) y_i y_j. \tag{12}$$

If the trajectory reaches the boundary of M, elastic reflection from the boundary occurs. The phase flow of this Hamiltonian system (with elastic collisions taken into account) preserves the Liouville measure $d\mu = d^n x\, d^n y$.

According to Gibbs, finding the system in a given state is a random event. The probability of this event at the initial instant is given by the probabilistic measure with a summable density $\rho(x, y)$,

$$\rho \in L_1(\Gamma, \mu), \qquad \int_\Gamma \rho\, d\mu = 1.$$

The phase flow g^t of the Hamiltonian system transfers this measure: at the instant t, its density is defined by the expression $\rho_t(z) = \rho \circ g^{-t}(z)$, where $z = (x, y)$ is a point of the phase space Γ.

Let $\varphi\colon M \to \mathbb{R}$ be a bounded measurable function on the configuration space. The function φ can be continued to a measurable function $\widetilde{\varphi}\colon \Gamma \to \mathbb{R}$ in accordance with the formula $\widetilde{\varphi}(x, y) = \varphi(x)$ for all $y \in T_x^* M$. We set

$$k(t) = (\rho \circ g^{-t}, \widetilde{\varphi}). \tag{13}$$

This function is well defined for all t (here, $p = 1$ and $q = \infty$). If φ is the characteristic function of some measurable domain $\Phi \subset M$, then $k(t)$ is the fraction of Hamiltonian systems in the Gibbs ensemble that are in the domain Φ at the instant t.

Theorem 3. *The limit $\lim_{t\to\pm\infty} k(t)$ exists and is given by $(\bar{\rho}, \varphi)$, where $\bar{\rho}$ is the Birkhoff average of the density ρ.*

This statement is a direct consequence of Theorem 2 and of stratification of the geodesic flow (with elastic reflections).

Corollary 1. *Let the flow g^t be ergodic at energy levels $H > 0$. Then Hamiltonian systems in the Gibbs ensemble are distributed on the configuration space $M^n = \{x\}$ with the density*

$$\left(\int_M (\det A)^{-1/2}\, d^n x\right)^{-1} \frac{d^n x}{(\det A)^{1/2}}, \qquad A = (a_{ij}(x)). \tag{14}$$

This density is a volume element on an n-dimensional Riemann manifold M whose metric is generated by the kinetic energy of system (12).

Proof. Indeed, let φ be the characteristic function of a measurable domain Φ on M. In view of ergodicity, the quantity $\bar\rho$ is a function of the energy $H = (Ay, y)/2$. Using the linear transformation $y = C(x)u$, we bring this positive-definite quadratic form to the sum of squares,

$$H = \frac{(C^T ACu, u)}{2} = \frac{(u, u)}{2}, \qquad C^T AC = E,$$

where E is the unit $n\times n$ matrix. Then

$$\int_\Gamma \bar\rho(H)\varphi\, d^n x\, d^n y = \int_\Gamma (\det A)^{-1/2}\bar\rho\left(\frac{u^2}{2}\right)\varphi\, d^n x\, d^n u = \alpha \int_\Phi (\det A)^{-1/2}\, d^n x,$$

where α is independent of Φ, which was to be proved.

We now consider the most interesting case from the standpoint of applications: M is a subdomain in $\mathbb{R}^n$ with the Euclidean metric, and H is a quadratic form in the momenta with constant coefficients. In particular, distribution density (14) is constant.

Corollary 2. *The limit distribution in M is uniform in $t \to \pm\infty$ if and only if the Birkhoff average $\bar\rho$ is independent of the point in M.*

More precisely, the last condition is to be understood such that there exists a function $\bar\rho'$ that is independent of x and coincides with $\bar\rho$ almost everywhere. The above uniform distribution criterion is satisfied by not only ergodic but also some integrable systems.

Example 5. We consider an ideal gas as a collisionless continuous medium in a rectangular parallelepiped $\Pi^n \subset \mathbb{R}^n$: each particle of the medium moves inertially, independently of other particles, reflecting elastically from the walls of Π^n. The claim is that independently of the initial distribution of gas particles over the volume Π and over velocities, the gas irreversibly tends to fill the volume Π uniformly as $t \to \pm\infty$. This observation by Poincaré [5] is rigorously proved in [6]. It can be simply derived from Corollary 2. Indeed, the limit density $\bar\rho$ is a first integral of the billiard in Π—a dynamical system with elastic impacts. This system is completely integrable and nondegenerate: it admits n independent first integrals given by squares of the particle momentum projections on the edges of the parallelepiped Π. Because of nondegeneracy, $\bar\rho$ is a function of only these integrals. Therefore, $\bar\rho$ is independent of a point in Π; hence (in accordance with Corollary 2), the limit gas distribution is uniform.

A more interesting and instructive example is as follows.

Example 6. We consider n identical particles on a segment $0 \le x \le a$; their coordinates $x_1, \ldots, x_n$ satisfy the inequalities

$$0 \le x_1 \le x_2 \le \cdots \le x_n \le a. \tag{15}$$

The particles collide elastically with each other and with the boundaries of the segment $\{x = 0\}$ and $\{x = a\}$. Because particles of the same mass exchange their velocities in elastic collisions, the particle velocity distribution is unchanged, and the system does not tend to the Maxwell distribution (contrary to common concepts of statistical mechanics, where additional Boltzmann assumptions are used). Yet any initial distribution density ρ specified in the $2n$-dimensional phase space of this system weakly converges to the Birkhoff average $\bar{\rho}$, which is independent of the coordinates $x_1, \ldots, x_n$. Therefore, in the statistical (thermal) equilibrium, all positions of n colliding particles have the same probability.

The above system with elastic impacts is in fact an integrable billiard in domain (15). Its complete integrability follows from the finiteness of the Coxeter group of polyhedron (15) (this is the group with the graph B_n; see, e.g., [7]). The corresponding billiard in (15) has n independent involutive integrals given by homogeneous polynomials in momenta with constant coefficients. It can be easily verified that this integrable system is nondegenerate.

Similar arguments also apply to a more general problem of the Boltzmann–Gibbs gas: the problem of the motion of a set of identical balls in an n-dimensional parallelepiped Π^n ($n \le 2$) colliding elastically with each other and with the walls of the vessel. It is assumed that if the radii of the balls are sufficiently small, this dynamical system with impacts is ergodic for each positive value of the total energy (see the discussion of this issue in [8], [9]). But Corollary 2 then implies that independently of the initial (sufficiently regular) probability distribution in the configuration space of the Boltzmann–Gibbs gas, the limit probability distribution is uniform. For the latter, substantial fluctuations of particle density have low probability, and this statement can therefore be interpreted such that the Boltzmann–Gibbs gas irreversibly tends to a uniformly filled volume. We emphasize that this conclusion does not involve a special Boltzmann assumption about the statistical independence of pair collisions of balls.

4. Ergodic theorems in the spaces L_p

The following ergodic theorem plays an important role below.

Theorem 4. *Let g^t be a flow on the space M with a measure μ, $\mu(M) < \infty$, and let $f \in L_p(M, \mu)$, $1 \le p < \infty$. Then convergence with respect to the L_p norm holds*:

$$\frac{1}{T} \int_0^T f \circ g^{-t}(z)\, dt \to \bar{f}(z). \tag{16}$$

This theorem is known and is in fact an almost obvious corollary of the Ionescu-Tulcea theorem [2] or of a more general theorem by Ackoglu [1] (also see [10]) stating that convergence in (16) occurs almost everywhere for $f \in L_p(M, \mu)$.

Proof. Indeed, for any predefined $\varepsilon_1, \varepsilon_2 > 0$, we can use this fact to find $T_0 > 0$ such that for all $T > T_0$,

$$\left| \frac{1}{T} \int_0^T f \circ g^{-t}(z)\, dt - \bar{f}(z) \right| < \varepsilon_1, \quad z \in M_{\varepsilon_1, \varepsilon_2},$$

where $\mu(M \setminus M_{\varepsilon_1, \varepsilon_2}) < \varepsilon_2$.

We choose ε_2 such that

$$\int_N |f|^p(z)\, d\mu(z) < \varepsilon_1^p$$

for any set $N \subset M$ with $\mu(N) < \varepsilon_2$. Then, obviously,

$$\int_{M \setminus M_{\varepsilon_1, \varepsilon_2}} |\bar{f}|^p(z)\, dz < \varepsilon_1^p.$$

For $T > T_0$, the quantity

$$\left\| \frac{1}{T} \int_0^T f \circ g^{-t}\, dt - \bar{f} \right\|_{L_p}^p$$

has the form

$$\left(\int_{M_{\varepsilon_1,\varepsilon_2}} + \int_{M \setminus M_{\varepsilon_1,\varepsilon_2}} \right) \left(\frac{1}{T} \int f \circ g^{-t}(z)\, dt - \bar{f}_p(z) \right)^p d\mu(z) \leq \varepsilon_1^p \mu(M) + 2^p \varepsilon_1^p,$$

which implies the statement of the theorem.

Remark. For $p = 1$, Theorem 4 coincides with the well-known result in the ergodic theory: for a space with a finite measure, convergence almost everywhere implies mean convergence (with respect to the L_1 norm; see, e.g., [11]). For $p = 2$, Theorem 4 coincides with the classical statistical Von Neumann theorem (but the latter holds without the assumption that the measure is finite).

5. Generalization of the statistical ergodic theorem

Theorem 2 contains Theorem 4 as a special case. Indeed, let h be the characteristic function of the segment $[0, 1]$. For any integrable function $\varphi\colon \mathbb{R} \to \mathbb{R}$, we then have

$$\int_{-\infty}^{+\infty} h(\omega)\varphi(\omega t)\, d\omega = \frac{1}{t} \int_0^t \varphi(s)\, ds.$$

The proof of Theorem 2 also involves Theorem 4. For any $\varepsilon > 0$, there exists a piecewise constant function $h_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ such that

1. $h_\varepsilon(\omega) = c_k = \text{const}$ on intervals (ω_k, ω_{k+1}), $k = 1, \dots, N$ (possibly, $\omega_1 = -\infty$ and/or $\omega_{N+1} = +\infty$, in which cases we respectively set $c_1 = 0$ and/or $c_2 = 0$),
2. $\Delta \subset (\omega_1, \omega_{N+1})$, and
3. $\int_\Delta |h - h_\varepsilon|\, d\omega < \varepsilon$.

Then (recalling that the measure μ is g^t-invariant)

$$\left| \int_\Delta h(\omega)(f_p \circ g^{-t}, f_q)\, d\omega - \int_\Delta h_\varepsilon(\omega)(f_p \circ g^{-t}, f_q)\, d\omega \right| \leq$$

$$\leq \int_\Delta |h - h_\varepsilon|\, d\omega\, \|f_p\|_{L_p} \|f_q\|_{L_q} \leq \varepsilon \|f_p\|_{L_p} \|f_q\|_{L_q}.$$

It therefore suffices to establish convergence of the integrals

$$J_k(t) = \int_{\omega_k}^{\omega_{k+1}} h_\varepsilon(\omega)(f_p \circ g^{-t}, f_q)\, d\omega.$$

In accordance with Theorem 4,

$$J_k(t) = \frac{c_k}{t} \int_{\omega_k t}^{\omega_{k+1} t} (f_p \circ g^{-t}, f_q)\, ds \to c_k(\omega_{k+1} - \omega_k)(\bar{f}_p, f_q)$$

as $t \to \infty$. It remains to note that

$$\sum_1^N c_k(\omega_{k+1} - \omega_k) = \int_\Delta h_\varepsilon(\omega)\, d\omega = \int_\Delta h(\omega)\, d\omega + \delta,$$

where $|\delta| \leq \varepsilon$, which was to be proved.

6. Stratified flows and evolution of measures

In this section, we prove Theorem 1. Let $f_p \in L_p(\Gamma, \mu)$ and $f_q \in L_q(\Gamma, \mu)$ with $\Gamma = \Lambda \times \Delta$. We set

$$f_{p,\omega}(\cdot) = f_p(\cdot, \omega), \qquad f_{q,\omega}(\cdot) = f_q(\cdot, \omega).$$

Applying the Fubini theorem and Eq. (9), we obtain

$$\int_\Gamma f_p \circ g^{-t}(z,\omega) f_q(z,\omega)\, d\mu(z)\, d\omega = \int_\Delta (f_{p,\omega} \circ g_\omega^{-t}, f_{q,\omega})\, d\sigma = \\ = \int_\Delta (f_{p,\omega} \circ g_*^{\alpha(\omega)t}, f_{q,\omega})\, d\sigma.$$

Let A_p and A_q be measurable subsets of Λ, and let Δ_p and Δ_q be intervals in Δ. Let $\chi_p, \chi_q \colon \Gamma \to \mathbb{R}$ be the characteristic functions (indicators) of the sets $A_p \times \Delta_p$ and $A_q \times \Delta_q$. We set

$$J(t) = \int_\Gamma \chi_p \circ g^{-t}(z,\omega) \chi_q(z,\omega)\, d^n z\, d\omega.$$

The Main Lemma. *The limits* $\lim_{t\to\pm\infty} J(t)$ *exist.*

Theorem 1 follows from the main lemma. Indeed, because ν_* is absolutely continuous with respect to the measure on Λ determined by some Riemannian metric and σ is absolutely continuous with respect to the Lebesgue measure, the space of continuous functions on Γ with a compact support is everywhere dense in $L_p(\Gamma, \mu)$. In turn, the linear space of functions that are finite linear combinations of indicators of μ-measurable subsets of Γ is everywhere dense in this space (even with respect to the C^0 norm).

Now let the functions $f_p \in L_p(\Gamma, \mu)$ and $f_q \in L_q(\Gamma, \mu)$ be arbitrary. To prove the existence of the limit

$$\lim_{t\to+\infty} (f_p \circ g^{-t}, f_q),$$

we use the Cauchy criterion: the difference

$$(f_p \circ g^{-t_1}, f_q) - (f_p \circ g^{-t_2}, f_q) \tag{17}$$

must be smaller than any fixed $\varepsilon > 0$ for all $t_1, t_2 > T(\varepsilon)$. For this, we approximate f_p and f_q in the corresponding norms by the functions φ_p and φ_q that are finite linear combinations of indicators: for any $\varepsilon > 0$, there exist functions φ_p and φ_q such that

$$\|f_p - \varphi_p\|_{L_p} < \varepsilon, \qquad \|f_q - \varphi_q\|_{L_q} < \varepsilon. \tag{18}$$

After this observation, the difference in (17) is to be represented as

$$(\varphi_p \circ g^{-t_1}, \varphi_q) - (\varphi_p \circ g^{-t_2}, \varphi_q) + ((f_p - \varphi_p) \circ g^{-t_1}, f_q - \varphi_q) + \\ + ((f_p - \varphi_p) \circ g^{-t_1}, \varphi_q) + (f_p \circ g^{-t_1}, f_q - \varphi_q) - ((f_p - \varphi_p) \circ g^{-t_2}, f_q - \varphi_q) - \\ - ((f_p - \varphi_p) \circ g^{-t_2}, \varphi_q) - (f_p \circ g^{-t_2}, f_q - \varphi_q). \tag{19}$$

In accordance with the main lemma, the difference of the first two terms can be made arbitrarily small for sufficiently large values of t_1 and t_2. In view of the Cauchy–Schwarz inequalities, the g^t-invariance of the measure μ, and inequality (18), the other terms in (19) tend to zero uniformly in t_1 and t_2 as $\varepsilon \to 0$.

Proof of the main lemma. We set $A_0 = A_p \cap A_q$ and $\Delta_0 = \Delta_p \cap \Delta_q$. Let $\chi_0 \colon \Lambda \times \Delta \to \mathbb{R}$ be the characteristic function (indicator) of the measurable set $A_0 \times \Delta_0$ and $\widetilde{\chi}_0 \colon \Lambda \to \mathbb{R}$ be the characteristic function of the set A_0. It is clear that

$$J(t) = \int_{\Delta_0} (\widetilde{\chi}_0 \circ g_*^{\alpha(\omega)t}, \widetilde{\chi}_0)\, d\sigma.$$

Let $D_\gamma = \{\omega \in \Delta_0 \colon |\alpha'(\omega)| > \gamma\}$, $\alpha' = d\alpha/d\omega$. In accordance with the assumption, the critical points of the function $\omega \mapsto \alpha(\omega)$ are isolated. Therefore, the σ-measure of the set $I \setminus D_\gamma$ tends to zero as $\gamma \to 0$. Moreover, we can assume that D_γ is a union of a finite number of intervals. Let (ω_1, ω_2) be one of the intervals comprising D_γ. Then α can be considered a coordinate on (ω_1, ω_2). Indeed, the function $\omega(\alpha)$ inverse to $\alpha \colon (\omega_1, \omega_2) \to \mathbb{R}$ exists and is smooth. We set $d\sigma(\omega) = h(\omega)\, d\omega$. In accordance with the conditions in Theorem 3, the function $\omega \to h(\omega)$ is integrable, $h \in L_1(I, d\omega)$. Therefore,

$$\int_{\omega_1}^{\omega_2} (\widetilde{\chi}_0 \circ g_*^{\alpha(\omega)t}, \widetilde{\chi}_0)\, h(\omega)\, d\omega = \int_{\alpha(\omega_1)}^{\alpha(\omega_2)} (\widetilde{\chi}_0 \circ g_*^{\alpha(\omega)t}, \widetilde{\chi}_0)\, h(\omega(\alpha))\, \omega'(\alpha)\, d\alpha. \tag{20}$$

Because $h(\omega(\alpha))\omega'(\alpha) \in L_1((\alpha(\omega_2), \alpha(\omega_1)), d\alpha)$, integral (20) has a limit as $t \to \infty$ in accordance with Theorem 2.

The lemma is completely proved.

Appendix: Proof of the proposition

Let $k(t) \to k_\infty$ as $t \to \infty$. Then by the Cauchy theorem,

$$\frac{1}{T} \int_0^T (\rho_t, \varphi)\, dt \to k_\infty$$

as $T \to \infty$. By the Fubini theorem, we have

$$\frac{1}{T} \int_0^T (f_p \circ g^{-t}, f_q)\, dt = \int_\Gamma \tilde{f}_p(z, T) f_q(z)\, d\mu(z),$$

where

$$\tilde{f}_p(z, T) = \frac{1}{T} \int_0^T f \circ g^{-t}(z)\, dt.$$

Next, Theorem 4 implies that

$$\int_\Gamma (\tilde{f}_p - \bar{f}_p)\, d\mu(z) \to 0$$

as $T \to \infty$. This implies that

$$\frac{1}{T} \int_0^T (f_p(\cdot, t), f_q)\, dt \to (\bar{f}_p, f_q).$$

Indeed,

$$\int_\Gamma (\tilde{f}_p(z, T) - \bar{f}_p(z)) f_q\, d\mu(z) \le \|\tilde{f}_p(\cdot, T) - \bar{f}_p\|_{L_p} \|f_q\|_{L_q} \to 0$$

as $T \to \infty$, which was to be proved.

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