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POLYNOMIAL CONSERVATION LAWS IN QUANTUM SYSTEMS

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We consider systems with a finite number of degrees of freedom and potential energy that is a finite sum of exponentials with purely imaginary or real exponents. Such systems include the generalized Toda chains and systems with a toric configuration space. We consider the problem of describing all the quantum conservation laws, i.e., the differential operators that are polynomial in the derivatives and commute with the Hamiltonian operator. We prove that in the case where the potential energy spectrum is invariant under reflection with respect to the origin, such nontrivial operators exist only if the system under consideration decomposes into a direct sum of decoupled subsystems. In the general case (without the spectrum symmetry assumption), we prove that the existence of a complete set of independent conservation laws implies the complete integrability of the corresponding classical system.

Keywords: Hamiltonian operator, polynomial differential operator, system with exponential interaction, potential spectrum

1. Main notation

Let $x = (x_1, \dots, x_n)$ be the generalized coordinates and $p = (p_1, \dots, p_n)$ be the conjugate canonical momenta of a classical Hamiltonian system. Let ϑ be a symmetric real $n\times n$ matrix. It corresponds to the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$ in the momentum space $\mathbb{R}^n = \{p\}$,

$$\langle p', p''\rangle = \sum_{j,l=1}^{n} \vartheta_{jl} p'_j p''_l, \quad p', p'' \in \mathbb{R}^n.$$

We set

$$\partial_j = \frac{\partial}{\partial x_j}, \qquad \partial = (\partial_1, \dots, \partial_n).$$

We consider the second-order differential operator $\Theta = \langle \partial, \partial \rangle$. If ϑ is the unit matrix, then $\Theta = \Delta$ is the Laplace operator. In what follows, we identify $-\Theta$ with the kinetic energy of some quantum system. For simplicity of notation, the Planck constant is set equal to one. In the general case, the matrix Θ is not assumed to be positive definite or even nondegenerate.

We also consider the potential energy $\hat{v}$, the operator of multiplication by the function

$$v = \sum_{k\in\mathfrak{M}} v^k e^{(k,x)},$$

where $\mathfrak{M} \subset \mathbb{C}^n$ is a finite set (the potential spectrum), $(k, x) = \sum k_j x_j$ is the operation of pairing a covector k and a vector x represented in dual bases, and v^k are complex numbers.

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We call the operator

$$\widehat{H} = -\Theta + \hat{v} \tag{1.1}$$

a Hamiltonian operator. It can be obtained from the classical Hamiltonian $\langle p, p\rangle + v$ by applying the quantization operation

$$x, p \mapsto \hat{x}, -\frac{i\partial}{\partial x}.$$

Our main problem is to find an operator

$$\widehat{F}(x, \partial) = \sum_{|\mu| \le M} f_\mu(x)\partial^\mu, \tag{1.2}$$

$$\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_+^n = \{0, 1, 2, \dots\},$$

$$|\mu| = \mu_1 + \dots + \mu_n, \qquad \partial^\mu = \partial_1^{\mu_1} \times \dots \times \partial_n^{\mu_n},$$

$$f_\mu = \sum_k f_\mu^k e^{(k,x)}, \tag{1.3}$$

commuting with $\widehat{H}$,

$$[\widehat{F}, \widehat{H}] = 0, \tag{1.4}$$

where $[\,\cdot\,,\,\cdot\,]$ is the commutator $[\hat{A}, \widehat{B}] = \hat{A} \circ \widehat{B} - \widehat{B} \circ \hat{A}$ and $\circ$ denotes composition of operators. An operator commuting with the Hamiltonian operator is called a first integral.

Hermitian differential operators are usually considered in quantum mechanics. It is well known that if $[\widehat{H}, \widehat{F}] = 0$, then the mean of the Hermitian operator $\widehat{F}$ in any state is independent of time,

$$\frac{d}{dt}\int \bar{\psi}\widehat{F}\psi\, dx_1 \cdots dx_n = 0,$$

for any wave function ψ satisfying the Schrödinger equation $i\partial\psi/\partial t = \widehat{H}\psi$. Therefore, any Hermitian differential operator commuting with the Hamiltonian operator generates a conservation law. In what follows, we consider the general problem of commuting operators and do not assume that the operator $\widehat{F}$ is Hermitian.

The following two cases are most interesting:

a. $\Theta = \Delta$, $\mathfrak{M} = -\mathfrak{M} \subset i\mathbb{Z}^n$, and $v^k = \bar{v}^{-k}$, which corresponds to a natural system on a torus (it can be assumed that $x \in \mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n)$), with the potential given by a trigonometric polynomial, and

b. $\Theta = \Delta$, $\mathfrak{M} \subset \mathbb{R}^n$, and $v^k \in \mathbb{R}$, where we deal with quantum generalized Toda chains.

Remark 1.1. A. In cases a and b, the operator $\widehat{H}$ is real.

B. Case a reduces to case b by the replacement

$$x \mapsto ix, \qquad \mathfrak{M} \mapsto i\mathfrak{M}, \qquad v \mapsto -v, \qquad \widehat{H} \mapsto -\widehat{H}.$$

Therefore, all objects are real in what follows.

C. By an orthogonal change of the coordinates x, the matrix ϑ can be brought to a diagonal form. Next, rescaling allows ensuring that the diagonal elements of ϑ are ± 1 or zeros. Therefore, in the case of a positive definite matrix ϑ, we can assume that $\Theta = \Delta$.

D. We are interested in nontrivial solutions of Eq. (1.4),

$$\widehat{F} \neq \sum_s c_s \widehat{H}^s, \quad s \in \{0, 1, \dots\}, \quad c_s = \text{const}\,. \tag{1.5}$$

A polynomial

$$F_{(M)} = \sum_{|\mu|=M} f_\mu(x) p^\mu$$

that is homogeneous in the p variables is called the principal symbol of the operator $\widehat{F}$.

2. The symmetric spectrum case

Definition 2.1. We say that a spectrum $\mathfrak{M}$ *separates* if $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$, $\mathfrak{M}_1 \perp \mathfrak{M}_2$, such that neither $\mathfrak{M}_1$ nor $\mathfrak{M}_2$ generates $\mathbb{R}^n$. Orthogonality is here understood with respect to the metric $\langle \cdot\,, \cdot \rangle$.

We note that this definition applies to a spectrum lying in a linear subspace of a positive codimension. In this case, one of the sets $\mathfrak{M}_j$ is empty.

If the spectrum separates, then the system has an additional integral quadratic in ∂. In this case, the Hamiltonian operator $\widehat{H}$ decomposes into the sum $\widehat{H}_1 + \widehat{H}_2$ of operators of form (1.1) that are Hamiltonian operators of separate subsystems of the original quantum system. Because $[\widehat{H}_1, \widehat{H}_2] = 0$, it then follows that $[\widehat{H}, \widehat{H}_1] = [\widehat{H}, \widehat{H}_2] = 0$.

Evidently, if one of the sets $\mathfrak{M}_j$ is empty, then the system has an integral linear in ∂.

Theorem 2.1. *Let the matrix ϑ be positive or negative definite, $\mathfrak{M} = -\mathfrak{M}$, and let there exist a nontrivial integral $\widehat{F}$ (in the sense of* (1.5)*). Then the spectrum separates.*

Corollary 2.1. *If Hamiltonian operator* (1.1) *with a positive or negative definite matrix ϑ and with a symmetric spectrum admits a nontrivial polynomial operator commuting with $\widehat{H}$, then there exists a nontrivial operator that is polynomial in ∂ and commutes with $\widehat{H}$, and its degree is at most two.*

Indeed, if the spectrum does not separate, then there are no nontrivial commuting operators at all. Vice versa, if the spectrum separates, then (as noted above) there is an operator that commutes with $\widehat{H}$ and is either quadratic or linear in the derivatives.

Apparently, a more general statement also holds. Hamiltonian operator (1.1) with a positive or negative definite matrix ϑ and a symmetric spectrum has $l \leq n$ independent integrals if and only if the spectrum $\mathfrak{M}$ separates into l orthogonal parts, i.e.,

$$\mathfrak{M} = \mathfrak{M}_1 \cup \dots \cup \mathfrak{M}_l, \qquad \mathfrak{M}_j \perp \mathfrak{M}_s, \quad j, s = 1, \dots, l,$$

such that none of the sets $\mathfrak{M} \setminus \mathfrak{M}_j$ generates $\mathbb{R}^n$.

Example 2.1. We apply these results to the quantum problem of n identical pairwise interacting particles moving along the circle $\{x \bmod 2\pi\}$. Their dynamics are described by Hamiltonian operator (1.1), where $\Theta = \Delta/(2\mu)$ (μ is the particle mass) and

$$v = \sum_{i<j} f(x_i - x_j). \tag{2.1}$$

We take the pair interaction potential f to be a trigonometric polynomial,

$$f(z) = \sum_{|m| \leq N} f_m e^{imz}, \quad f_N \neq 0, \quad N \geq 1. \tag{2.2}$$

We can consider that the particles move along the line $\mathbb{R} = \{x\}$ with a periodic pair potential.

It is clear that in view of the translational symmetry, this system always admits the momentum integral $\widehat{\Phi} = \sum \partial_j$ in addition to the total energy $\widehat{H}$. We can therefore speak of conditions for the existence of an additional polynomial operator independent of $\widehat{H}$ and $\widehat{\Phi}$ and commuting with these operators. Using the momentum integral, we could reduce the number of degrees of freedom by one (passing, for example, to the barycentric system of coordinates). But we do not do this, partly because we wish to preserve the symmetry of the Hamiltonian operator with respect to the coordinates $x_1, \dots, x_n$.

The spectrum $\mathfrak{M}$ of potential (2.1) consists of points of the form $(0, \dots, l, \dots, -l, \dots, 0)$, where $|l| \le N$. All these points lie in a hyperplane Π passing through the origin. This property is a manifestation of the existence of a linear integral $\widehat{\Phi}$.

It is easy to see that $n-1$ points of the spectrum

$$(N, -N, 0, \dots, 0), \quad (N, 0, -N, 0, \dots, 0), \quad \dots, \quad (N, 0, 0, \dots, -N)$$

are linearly independent (as vectors in $\mathbb{Z}^n$) and their pairwise scalar products are positive. Therefore, the spectrum $\mathfrak{M}$ does not separate on the hyperplane Π, and in accordance with Theorem 2.1, the system of interacting particles under consideration does not admit new polynomial operators commuting with $\widehat{H}$ and $\widehat{\Phi}$.

We note that in its classical version, the problem of the motion of interacting particles with the potential in (2.1), (2.2) also does not admit an additional integral given by a polynomial in the momenta, independent of the energy and total momentum integrals. This was established in [1] (Appendix 6) using the methods developed in [2] based on the classical perturbation theory approach.

It was hypothesized in [3] that Theorem 2.1 and Corollary 2.1 also hold for classical multidimensional Hamiltonian systems. Based on the method in [2], sufficient conditions (in terms of the geometry of the convex hull of the spectrum) for the absence of an additional first integral polynomial in the momenta were also given in [3]. The method developed in what follows seems to allow proving the hypothesis in [3] in full.

The proof of Theorem 2.1 consists of two parts, analytic (Secs. 4–7) and geometric (Secs. 8–10). The analytic part consists in the analysis of Eq. (1.4). Its principal result is the proof of the main lemma formulated below. The geometric part consists in using the main lemma to prove Theorem 2.1. These two parts are independent, and the reader may first read the proof of the theorem and then read the proof of the main lemma.

We now formulate the main lemma.

Definition 2.2. We call $\alpha \in \mathfrak{M}$ a *vertex of the spectrum* if for any positive integer j and any $k_1, \dots, k_j \in \mathfrak{M}$, the equality $j\alpha = k_1 + \dots + k_j$ is possible only for $k_1 = \dots = k_j = \alpha$.

Obviously, if $\alpha \ne 0$ is a vertex of a polyhedron given by the convex hull of $\mathfrak{M} \cup \{0\}$, then α is a vertex of $\mathfrak{M}$.

Definition 2.3. Let $\alpha \in \mathfrak{M}$ be a vertex of the spectrum. A vector $\beta \in \mathfrak{M}$, $\beta \ne \alpha$, is said to be *adjoint* to α if for any $j \in \mathbb{N}$ and any $k_0, k_1, \dots, k_j \in \mathfrak{M}$, the equality $j\alpha + \beta = k_0 + \dots + k_j$ is possible only in the case where one of the vectors k_l, $l = 0, \dots, j$, is equal to β and the other vectors are equal to α.

With each vector $k \in \mathbb{R}^n$, we associate the differential operator

$$D_k = \left(k, \frac{\partial}{\partial p}\right) \equiv \sum_{j=1}^{n} k_j \frac{\partial}{\partial p_j}.$$

Let $G(\vartheta) \subset GL(n, \mathbb{R})$ be the group of orientation-preserving transformations of $\mathbb{R}^n$ that also preserve the metric ϑ (in general, indefinite). If ϑ is the standard Euclidean metric, then $G(\vartheta) = SO(n)$.

Main Lemma. *Let α and β be a linearly independent vertex and the vector adjoint to it such that*

$$-2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\notin\mathbb{Z}_+. \tag{2.3}$$

Let $g(\alpha,\beta)\subset G(\vartheta)$ be the group of all transformations in $G(\vartheta)$ that coincide with the identity on the vectors orthogonal to the plane spanned by α and β. Then $F_{(M)}$ is preserved under the action of $g(\alpha,\beta)$ on $\mathbb{R}^n$.

If α is an isotropic vector $\big(\langle\alpha,\alpha\rangle=0\big)$, then condition (2.3) must be replaced with the condition $\langle\alpha,\beta\rangle\neq 0$.

Corollary 2.2. *Let the spectrum contain α and β satisfying the condition of the main lemma. Let $\widehat{F}^{(1)},\dots,\widehat{F}^{(n)}$ be first integrals. Then their principal symbols $F^{(1)}_{(M_1)},\dots,F^{(n)}_{(M_n)}$ are identically dependent.*

The proof of corollary 2.2 is based on two simple facts.

1. The principal symbol of a first integral is independent of x (see Proposition 4.2 below).
2. The derivative of any function $F^{(j)}_{(M_j)}=F^{(j)}_{(M_j)}(p)$ along a vector field tangent to orbits of the action of $g(\alpha,\beta)$ on $\mathbb{R}^n$ is equal to zero.

3. Generalized Toda chains with a complete set of commuting operators

Corollary 2.2 of the main lemma allows comprehensively classifying quantum n-dimensional systems with an exponential interaction (such systems, in particular, include Toda chains and their generalizations [4]) that can admit n pairwise commuting differential operators that are polynomial in ∂ and whose principal symbols are functionally independent. Such sets of operators can be called complete. The point is that there exist no new operators commuting with the Hamiltonian operator whose principal symbols are independent of the principal symbols of the already existing n commuting operators. Indeed, the principal symbol of an operator commuting with the Hamiltonian operator depends on only n variables $p_1,\dots,p_n$ (see Proposition 4.2 below).

The classical analogue of the above problem has been widely discussed in the literature. In addition to [4] mentioned above, we also note [5], where a number of integrable generalizations of the standard Toda chains were obtained. The ultimate classification of classical integrable generalized Toda chains was obtained in [6].

The problem of finding quantum systems admitting a complete set of independent integrals (conservation laws) is well known (see, e.g., [7] and the references therein). It is natural to regard such systems as rare and, in a certain sense, more regular compared with generic systems.

We have the following theorem.

Theorem 3.1. *Let a quantum system with an exponential interaction have a complete set of pairwise commuting differential operators that are polynomial in ∂. Then the corresponding classical system is integrable, i.e., has a complete set of pairwise commuting first integrals that are polynomial in the momenta.*

The sought classification is based on applying the relations

$$-2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\in\{0,1,2,\dots\}, \tag{3.1}$$

where α is *any* vertex of the system spectrum and β is the vector adjoint to it (see the main lemma). The problem of describing spectra $\mathfrak{M}$ with property (3.1) was solved in [6] in relation to classifying completely integrable classical Hamiltonian systems with exponential interactions. The discrete series of possible spectra for irreducible systems with $n\geq 2$ degrees of freedom are determined by their *Dynkin diagrams*. These diagrams, given in [6], can be obtained from simple root diagrams of graded Kač–Moody algebras by adding vectors directed along some of the already existing vectors. The added vectors have either twice or half the length of the root vectors with the same direction.

The question of the *sufficiency* of conditions (3.1) for the existence of a complete set of independent polynomial commuting operators of quantum systems with an exponential interaction remains open. For the usual Toda chains, it is solved affirmatively.

Example 3.1. A closed three-particle quantum Toda chain admits three independent self-adjoint commuting operators

$$\widehat{H} = -\frac{1}{2}(\partial_1^2 + \partial_2^2 + \partial_3^2) + e^{x_1-x_2} + e^{x_2-x_3} + e^{x_3-x_1},$$

$$\hat{I} = i(\partial_1 + \partial_2 + \partial_3),$$

$$\widehat{F} = i\partial_1\partial_2\partial_3 + ie^{x_2-x_3}\partial_1 + ie^{x_3-x_1}\partial_2 + ie^{x_1-x_2}\partial_3.$$

The relations $[\widehat{H}, \hat{I}] = [\widehat{F}, \hat{I}] = [\widehat{H}, \widehat{F}] = 0$ are verified by direct calculation.

We note that the full spectral theory is presently elaborated only for *open* Toda chains (see [8] and also [9]).

4. Symbols of differential operators

We call (1.2) the *standard form* of a differential operator. A characteristic feature of this form is that all the derivatives are moved to the right. Evidently, any differential operator with smooth coefficients can be represented in the standard form.

With any operator $\widehat{F}$ of form (1.2), we associate its symbol,

$$\widehat{F}(x,\partial) \mapsto F(x,p) = \operatorname{symb}\big(\widehat{F}(x,\partial)\big), \quad p \in \mathbb{C}^n,$$

where F is a polynomial in p of the form

$$F(x,p) = \sum_{|\mu|\leq M} f_\mu(x) p^\mu. \tag{4.1}$$

In the literature, F is often called the left symbol (or xp-symbol) of $\widehat{F}$.

The operation symb is linear. For any operators $\widehat{F}$, $\widehat{G}$, $\widehat{F}_0$, and $\widehat{G}_0$, a function $u(x)$, and a vector $\mu \in \mathbb{Z}_+^n$, we have

$$\begin{aligned}
&\operatorname{symb}(\widehat{F} + \widehat{F}_0) \circ (\widehat{G} + \widehat{G}_0) = \operatorname{symb}(\widehat{F} \circ \widehat{G}) + \operatorname{symb}(\widehat{F} \circ \widehat{G}_0) + \operatorname{symb}(\widehat{F}_0 \circ \widehat{G}) + \operatorname{symb}(\widehat{F}_0 \circ \widehat{G}_0),\\
&\operatorname{symb}(u\widehat{F} \circ \widehat{G}) = u \operatorname{symb}(\widehat{F} \circ \widehat{G}),\\
&\operatorname{symb}(\widehat{F} \circ \widehat{G} \circ \partial^\mu) = \operatorname{symb}(\widehat{F} \circ \widehat{G}) p^\mu.
\end{aligned} \tag{4.2}$$

Proposition 4.1. *Let $\widehat{F}$ and $\widehat{G}$ be two operators, and let $F(x,p)$ and $G(x,p)$ be their symbols. Then*

$$\operatorname{symb}(\widehat{F}\circ\widehat{G}) = F(x,p+\partial)G(x,p). \tag{4.3}$$

Remark 4.1. In more detail, the right-hand side of (4.3) can be written as

$$\sum_{\mu} f_\mu(x)(p+\partial)^\mu G(x,p).$$

Proof of Proposition 4.1. In accordance with relations (4.2), it suffices to consider the case where $\widehat{F}=\partial^\mu$ and $\widehat{G}=\hat{u}(x)$ with $u(x)$ being a smooth function:

$$\operatorname{symb}(\partial^\mu\circ\hat{u}) = (p+\partial)^\mu u.$$

This is proved by induction on $|\mu|$.

Corollary 4.1. *We have the equalities*

$$\operatorname{symb}[\widehat{F},-\Theta] = \big(2\langle p,\partial\rangle+\Theta\big)F(x,p),$$

$$\operatorname{symb}[\widehat{F},\hat{v}] = \big(F(x,p+\partial)-F(x,p)\big)v(x).$$

Corollary 4.2. *Equation* (1.4) *is equivalent to the equation*

$$\big(2\langle p,\partial\rangle+\Theta\big)F(x,p) + \big(F(x,p+\partial)-F(x,p)\big)v(x) = 0. \tag{4.4}$$

This is a partial differential equation in x. The task is to find its solution of form (4.1). We set

$$F_{(m)} = \sum_{|\mu|=m} f_\mu(x)p^\mu.$$

Then $F=\sum_{m=0}^{M}F_{(m)}$.

The mth-degree homogeneous form in the expansion of (4.4) in p is given by

$$2\langle p,\partial\rangle F_{(m-1)} + \Theta F_{(m)} + \sum_{r>0} F^{(r)}_{(m+r)}(x,p)(\partial)v(x) = 0, \tag{4.5}$$

where $F_{(j)}$ is considered equal to zero if $j\notin\{0,1,\dots,M\}$ and $F^{(r)}_{(s)}(\partial)$ is a homogeneous form of degree r in ∂ arising in the Taylor expansion of $F_{(s)}(x,p+\partial)-F_{(s)}(x,p)$ in the formal variable ∂ at the point $\partial=0$.

For any $k\in\mathbb{R}^n$, we define its order, $\operatorname{ord} k$, with respect to the spectrum $\mathfrak{M}$. The order of a vector k is the smallest $J\in\mathbb{Z}_+\cup\{+\infty\}$ such that

$$k = k_1+\cdots+k_J$$

for some $k_1,\dots,k_J\in\mathfrak{M}$. The order of zero is set equal to zero.

Remark 4.2. A. Because the spectrum is finite, $\operatorname{ord} k=+\infty$ for almost all $k\in\mathbb{R}^n$.

B. Let α be a vertex of the spectrum. Then for any $j\in\mathbb{N}$, we have $\operatorname{ord}(j\alpha)=j$.

C. Let α be a vertex of the spectrum and β be the vector adjoint to it. Then for any $j\in\mathbb{N}$, we have $\operatorname{ord}\big((j-1)\alpha+\beta\big)=j$.

Let Π_j be the space of functions of the form

$$\Psi(p,k) = \sum_{\operatorname{ord} k=j} \psi_k(p)e^{(k,x)}.$$

We set $\widetilde{\Pi}_j=\bigcup_{l\le j}\Pi_l$ and define the natural projection $\pi_j\colon\widetilde{\Pi}_j\to\Pi_j$.

Proposition 4.2. *Let F be the symbol of a first integral. Then*

$$F_{(M-s)} \in \widetilde{\Pi}_{[s/2]}, \tag{4.6}$$

where $[\,\cdot\,]$ denotes the integer part of a number. In particular, $F_{(M)}$ is independent of x.

Moreover, the functions

$$\Phi_{(M-s)} = \pi_{[s/2]} F_{(M-s)} \in \Pi_{[s/2]}$$

with even $s = 2j$ satisfy the recursive relations

$$2\langle p, \partial\rangle \Phi_{(M-2j-2)} + \pi_j \Biggl(\sum_{l=1}^{n} \frac{\partial}{\partial p_l} \Phi_{(M-2j)}(x,p) \partial_{x_l} v(x) \Biggr) = 0. \tag{4.7}$$

Proof. We first verify (4.6) using induction on s. For $m = M+1$, formula (4.5) has the form $2\langle p, \partial\rangle F_{(M)}(p,x) = 0$, whence in view of (1.3) and (4.1), we obtain $F_{(M)} = F_{(M)}(p)$. Therefore, formula (4.6) is satisfied for $s = 0$.

We now assume that (4.6) holds for all $s = 0, 1, \ldots, s_0$. In accordance with (4.5),

$$2\langle p, \partial\rangle F_{(M-s_0-1)} = -\Theta F_{(M-s_0)} - \sum_{r>0} F^{(r)}_{(M-s_0+r)}(x,p)(\partial)v(x). \tag{4.8}$$

Let s_0 be even. Then

$$F_{(M-s_0)} \in \widetilde{\Pi}_{s_0/2}, \qquad F^{(r)}_{(M-s_0+r)} \in \widetilde{\Pi}_{[(s_0-r)/2]} \subset \widetilde{\Pi}_{s_0/2-1}, \quad r > 0.$$

In view of the equality $\bigl[(s_0+1)/2\bigr] = s_0/2$, we obtain (4.6).

Now let s_0 be odd. Then

$$F_{(M-s_0)} \in \widetilde{\Pi}_{(s_0-1)/2}, \qquad F^{(r)}_{(M-s_0+r)} \in \widetilde{\Pi}_{[(s_0-r)/2]} \subset \widetilde{\Pi}_{(s_0-1)/2}, \quad r > 0.$$

Therefore, $F_{(M-s_0-1)} \in \widetilde{\Pi}_{(s_0+1)/2} = \widetilde{\Pi}_{[(s_0+1)/2]}$.

We obtain Eqs. (4.7) by applying the operator π_j to (4.8), where $s_0 + 1 = 2j + 2$.

5. Vertices and their adjoint vectors

5.1. Let α be a vertex of the spectrum. Then (in accordance with Proposition 4.2) for $j \geq 1$,

$$2\langle p, \partial\rangle \bigl(F^{j\alpha}_{(M-2j)} e^{j(\alpha,x)}\bigr) = -\sum_{l} \frac{\partial}{\partial p_l} \bigl(F^{(j-1)\alpha}_{(M-2j+2)} e^{(j-1)(\alpha,x)}\bigr) \frac{\partial}{\partial x_l} \bigl(v^{\alpha} e^{(\alpha,x)}\bigr), \tag{5.1}$$

where $F^{k}_{(s)} = F^{k}_{(s)}(p)$ is the coefficient of the function $F_{(s)}$ at $e^{(k,x)}$. This readily implies the following proposition.

Proposition 5.1. *Let $\alpha \in \mathfrak{M}$ be a vertex. Then for $j = 1, 2, \ldots, [(M+1)/2]$,*

$$F^{j\alpha}_{(M-2j)} = \frac{(v^{\alpha})^j}{(-2)^j j!} A^j F_{(M)}, \tag{5.2}$$

where the operator A has the form

$$A = \frac{1}{\langle p, \alpha\rangle} D_{\alpha}, \qquad D_{\alpha} = \Bigl\langle \alpha, \frac{\partial}{\partial p} \Bigr\rangle.$$

Expressions (5.2) must be polynomials in p. This imposes essential restrictions on $F_{(M)}$.

5.2. Similarly, using induction and Eq. (5.1), we can easily prove the following proposition.

Proposition 5.2. *Let $\alpha \in \mathfrak{M}$ be a vertex and β be the vector adjoint to α. Then for $j = 1, 2, \dots,$ $[(M+1)/2]$,*

$$F_{(M-2j)}^{(j-1)\alpha+\beta} = \frac{(v^\alpha)^{j-1} v^\beta}{(-2)^j \langle p, (j-1)\alpha + \beta \rangle} G_{j-1} F_{(M)}, \tag{5.3}$$

where the operators G_j are determined by the recursive relations

$$G_0 = D_\beta, \qquad G_j = D_\alpha \frac{1}{\langle p, (j-1)\alpha + \beta \rangle} G_{j-1} + D_\beta \frac{1}{j!} A^j, \quad j \in \mathbb{N}. \tag{5.4}$$

Expressions (5.3) must be polynomial in p. This imposes additional restrictions on $F_{(M)}$.

5.3. In what follows, we also need the following simple observation.

Proposition 5.3. *Let expressions (5.2) and (5.3) be polynomial in p for $j = 1, 2, \dots, [(M+1)/2]$. Then for integer $j > [(M+1)/2]$, the right-hand sides of (5.2) and (5.3) are identically equal to zero.*

Proof. The degree d of polynomials (5.2) and (5.3) is equal to $d = M - 2j$. For $j = [(M+1)/2]$, we have

$$d = \begin{cases} 0 & \text{if } M \text{ is even,} \\ -1 & \text{if } M \text{ is odd.} \end{cases}$$

This implies that expressions (5.2) and (5.3) are equal to zero for odd M and $j = [(M+1)/2]$ and are independent of p for even M. It remains to use Eq. (5.4).

5.4. The following lemma plays an important role below.

Technical Lemma. *We subject the operators G_s to the following actions. We first represent them in the standard form, i.e., as a sum of monomials of the form*

$$q(p) D_\alpha^l \qquad \text{or} \qquad q_*(p) D_\alpha^l D_\beta$$

(with all derivatives moved to the right). We then set $\langle p, s\alpha + \beta \rangle = 0$. The resulting operators are denoted by G_s^0. Then

$$G_s^0 = \frac{(-1)^s}{s! \, \langle p, \alpha \rangle^{2s}} \prod_{j=0}^{s-1} \left\langle \alpha, \beta + \frac{j}{2}\alpha \right\rangle D_{s\alpha+\beta}. \tag{5.5}$$

For $s = 0$, the product in (5.5) must be set equal to one.

An amazing fact is that in the calculation of G_s^0, the differentiation operators of an order greater than one cancel each other, and the first-order differentiation operators reduce to $D_{s\alpha+\beta}$ with some coefficient.

The technical lemma is proved in Sec. 7.

6. Proof of the main lemma

Lemma 6.1. *Let $\alpha \in \mathfrak{M}$ be a vertex of the spectrum, and let $\beta \in \mathfrak{M}$ be the vector adjoint to it such that conditions (2.3) are satisfied. Then*

$$D_{j\alpha+\beta} F_{(M)}(p)|_{\langle p, j\alpha+\beta \rangle = 0} = 0, \quad j \in \mathbb{Z}_+. \tag{6.1}$$

Proof. In accordance with Proposition 5.2, expressions (5.3) are polynomials. Therefore, for $s = 0, 1, \dots, [(M-1)/2]$,

$$\langle p, s\alpha + \beta\rangle \quad \text{divides} \quad G_s F_{(M)}(p). \tag{6.2}$$

With Proposition 5.3, conditions (6.2) are satisfied for all $s \in \mathbb{Z}_+$.

Applying the technical lemma, we find that

$$\langle p, s\alpha + \beta\rangle \quad (s \in \mathbb{Z}_+) \qquad \text{divides} \qquad D_{s\alpha+\beta} F_{(M)}(p) \tag{6.3}$$

We here use the fact that in accordance with (2.3), the coefficients of $D_{s\alpha+\beta}$ in (5.5) are nonzero.

We now prove the main lemma. Let L be the two-dimensional plane spanned by the vectors α and β. Then $\mathbb{R}^n = L \oplus L^\perp$. We fix an orthogonal projection $p_\perp$ of the vector p on $L^\perp$. Then the set of solutions of the equation $\langle p, a\alpha + b\beta\rangle = 0$ is the straight line $l_{a,b}$ parallel to L and passing through the point $p = p_\perp$. The set $\mathbf{l}$ of lines $l_{a,b}$ is isomorphic to the one-dimensional real projective space with the homogeneous coordinates a, b.

The set $Z \subset \mathbf{l}$ of points a, b for which

$$D_{a\alpha+b\beta} F_{(M)}(p)\big|_{p\in l_{a,b}} = 0, \tag{6.4}$$

is an algebraic submanifold in $\mathbf{l}$. In accordance with Lemma 6.1, the set Z contains infinitely many points of the form $b/a \in \mathbb{Z}_+$. Therefore, Z coincides with $\mathbf{l}$. Thus, (6.4) is satisfied for any a, b, and $p_\perp$.

For any p, the vector $v(p) = -\langle p, \beta\rangle\alpha + \langle p, \alpha\rangle\beta$ is orthogonal to p, and therefore

$$D_{v(p)} F_{(M)} \equiv 0.$$

It remains to note that the group $g(\alpha, \beta)$ is isomorphic to the phase flow generated by the vector field $v(p)$.

7. Proof of the technical lemma

We first reformulate the lemma in a more convenient form. Namely, we replace $\widetilde{\beta} = \beta + s\alpha$ in the operators G_s, which then become

$$G_0 = D_{\tilde\beta}, \qquad G_s = \sum_{l=0}^{s} B_l \circ D_{\tilde\beta - j\alpha} \frac{1}{(j-l)!} \circ A^{j-l}, \quad s > 0, \tag{7.1}$$

where

$$B_0 = 1, \qquad B_l = D_\alpha \circ \frac{1}{\langle p, \tilde\beta - \alpha\rangle} D_\alpha \circ \frac{1}{\langle p, \tilde\beta - 2\alpha\rangle} \cdots D_\alpha \circ \frac{1}{\langle p, \tilde\beta - l\alpha\rangle}, \quad l > 0.$$

The technical lemma can now be formulated as follows.

Lemma 7.1. *We subject the operators G_s to the following actions. We first represent them in the standard form, then set $\langle p, \tilde\beta\rangle = 0$, and let G_s^0 denote the resulting operators. Then*

$$G_s^0 = \frac{(-1)^s}{s!\langle p, \alpha\rangle^{2s}} \prod_{j=0}^{s-1} \left\langle \alpha, \tilde\beta - \left(s - \frac{j}{2}\right)\alpha \right\rangle D_{\tilde\beta}.$$

Proof. For brevity, we omit the tildes over β in what follows. We set

$$a = \langle \alpha, \alpha\rangle, \qquad b = \langle \alpha, \beta\rangle,$$

$$q_0 = 1, \qquad q_s = \prod_{j=1}^{s} \left(\frac{j}{2}a - b\right), \quad s \in \mathbb{N}.$$

We need the following propositions.

Proposition 7.1. *We take the operators B_l to the standard form and set $\langle p,\beta\rangle = 0$. Then the resulting operators are given by*

$$B_l^0 = \sum_{j=0}^{l} \frac{(-1)^{l-j}}{j!\,(l-j)!\,\langle p,\alpha\rangle^{2j}} \frac{q_{2j}}{q_j} A^{l-j}.$$

Proposition 7.2. *We take the operators $B_l D_\alpha$ to the standard form and set $\langle p,\beta\rangle = 0$. Then the resulting operators are given by*

$$B_l^+ = \frac{(-1)^l}{l!}\langle p,\alpha\rangle A^{l+1} + \sum_{j=0}^{l-1} \frac{(-1)^{l-j} b}{(j+1)!\,(l-j-1)!\,\langle p,\alpha\rangle^{1+2j}} \frac{q_{2j+1}}{q_{j+1}} A^{l-j}.$$

Propositions 7.1 and 7.2 are proved by induction.

For any $j \in \mathbb{Z}$,

$$D_\beta \circ \langle p,\alpha\rangle^j = \frac{b}{a} D_\alpha \circ \langle p,\alpha\rangle^j - \frac{b}{a}\langle p,\alpha\rangle^j D_\alpha + \langle p,\alpha\rangle^j D_\beta,$$

and the equalities

$$D_\beta \circ A^l = \frac{b}{a} D_\alpha \circ A^l - \frac{b}{a} A^l \circ D_\alpha + A^l \circ D_\beta, \quad l = 0, 1, \ldots, \tag{7.2}$$

therefore hold.

With identity (7.2) taken into account, Eqs. (7.1) can be rewritten as

$$G_j = \sum_{l=0}^{j} \frac{1}{(j-l)!} \left(\frac{b-ja}{a} B_l D_\alpha A^{j-l} - \frac{b}{a} B_l \circ A^{j-l} \circ D_\alpha + B_l \circ A^{j-l} \circ D_\beta \right).$$

Therefore,

$$G_s^0 = R_1 + R_2 + R_3,$$

where

$$R_1 = \sum_{l=0}^{s} \frac{1}{(s-l)!} \frac{b-sa}{a} B_l^+ \circ A^{s-l}, \qquad R_2 = -\sum_{l=0}^{s} \frac{1}{(s-l)!} \frac{b}{a} B_l^0 \circ A^{s-l} \circ D_\alpha,$$

$$R_3 = \sum_{l=0}^{s} \frac{1}{(s-l)!} B_l^0 \circ A^{s-l} \circ D_\beta.$$

The proof of the lemma is completed by applying the following proposition.

Proposition 7.3. *The equalities*

$$R_1 = -R_2 = \frac{b}{a} \frac{1}{s!\,\langle p,\alpha\rangle^{2s}} \frac{q_{2s}}{q_s} D_\alpha, \qquad R_3 = \frac{1}{s!\langle p,\alpha\rangle^{2s}} \frac{q_{2s}}{q_s} D_\beta$$

hold.

Proposition 7.3 is proved by direct computation using Propositions 7.1 and 7.2. For example, we verify the last equality as

$$R_3 = \sum_{l=0}^{s} \sum_{j=0}^{l} \frac{1}{(s-l)!} \frac{(-1)^{l-j}}{j!\,(l-j)!\,\langle p,\alpha\rangle^{2j}} \frac{q_{2j}}{q_j} A^{s-j} D_\beta =$$

$$= \sum_{j=0}^{s} \frac{1}{j!\,\langle p,\alpha\rangle^{2j}} \frac{q_{2j}}{q_j} A^{s-j} D_\beta \sum_{l=j}^{s} \frac{(-1)^{l-j}}{(s-l)!\,(l-j)!}.$$

Because the inner sum (over l) is nonvanishing only for $j = s$, we must set $j = l = s$, which implies the required formula for R_3.

8. Adjoint vector lemmas

This section opens the geometric part of the proof. Everywhere in what follows, we set

$$\mathfrak{C} = \operatorname{conv}\bigl(\mathfrak{M} \cup \{0\}\bigr),$$

where $\operatorname{conv}(S)$ is the convex hull of a set S.

In addition, because the matrix ϑ is positive or negative definite in accordance with the conditions in Theorem 2.1, it can be assumed that $\vartheta = \pm E_n$, where E_n is the unit matrix of size n. With the possibility of replacing $\widehat{H}$ with $-\widehat{H}$ in mind, we can assume that $\vartheta = E_n$. We therefore assume in what follows that $\langle\,\cdot\,,\,\cdot\,\rangle$ is the standard Euclidean scalar product and the group $G(\vartheta)$, preserving the metric ϑ, is $SO(n)$.

Lemma 8.1. *Let α be a vertex of the polyhedron $\mathfrak{C}$ and I be its edge containing α. Then I contains the vector $\beta \in \mathfrak{M}$ adjoint to α.*

Proof. We take β to be the element of the set $I \setminus \{\alpha\}$ that is nearest α. Let

$$k_1 + \cdots + k_{s+1} = s\alpha + \beta, \quad k_1, \ldots, k_{s+1} \in \mathfrak{M}. \tag{8.1}$$

We show that one of the vectors k_j is equal to β. Let $\Lambda \subset \mathbb{R}^n$ be the hyperplane such that $\Lambda \cap \mathfrak{C} = I$. Such a hyperplane exists because the polyhedron $\mathfrak{C}$ is convex. Then for any $k \in \mathfrak{M}$, the projection of k on the normal ν to Λ is less than or equal to the projection of α on ν, and the equality holds if and only if $k \in I$.

Projecting (8.1) on ν, we obtain $k_1, \ldots, k_{s+1} \in I$. Next, projecting (8.1) on I, we obtain the desired statement.

Definition 8.1. An edge I with the endpoints α_1 and α_2 is said to be *positive* if $\langle \alpha_1, \alpha_2 \rangle > 0$.

Corollary 8.1. *Let α and β be as in Lemma* 8.1. *If I is positive, then $\langle \alpha, \beta \rangle > 0$.*

Let $\nu_1, \ldots, \nu_m$ be some basis in $\mathbb{R}^n$. Then the lexicographic ordering $\prec$ is defined in $\mathbb{R}^n$. For a vector $u = \sum u_j \nu_j \in \mathbb{R}^n$, we say that $0 \prec u$ if the first nonzero number in the sequence $u_1, u_2, \ldots, u_n$ is positive. We say that $u \prec v$ if $0 \prec v - u$. Obviously, the relation $\prec$ thus defined agrees with the linear structure on $\mathbb{R}^n$:

$$\begin{aligned} &\text{if } u \prec v, && \text{then } u + w \prec v + w, \\ &\text{if } u \prec v \text{ and } \quad \lambda > 0, && \text{then } \lambda u \prec \lambda v. \end{aligned}$$

We consider the lexicographic ordering $\prec$ on $\mathbb{R}^n$ corresponding to some basis $e_1, e_2, \ldots, e_n$.

Lemma 8.2. *Let a vector α be maximal with respect to $\prec$ in the spectrum $\mathfrak{M}$. Then α is a vertex.*

Let $\Lambda \subset \mathbb{R}^n$ be a linear subspace containing α, and let β be the maximal vector among the vectors $k \in \mathfrak{M}$ such that $k \notin \Lambda$. Then β is adjoint to α.

Proof. Let

$$k_1 + \cdots + k_s = s\alpha, \quad k_1, \ldots, k_s \in \mathfrak{M}.$$

Because $k_j \preccurlyeq \alpha$, $j = 1, \ldots, s$, we have $k_1 = \cdots = k_s = \alpha$.

We now assume that Eqs. (8.1) is satisfied. Its right-hand side is not in Λ. Therefore, one of the vectors k_j (which we can assume to be k_1) is not in Λ. Then $k_1 \preccurlyeq \beta$, and $k_l \preccurlyeq \alpha$, $l = 2, \ldots, s$. This implies that $k_1 = \beta$.

Definition 8.2. We say that a vertex α of the polyhedron $\mathfrak{C}$ is *tame* if α is the unique intersection point of $\mathfrak{C}$ with the plane that is orthogonal to the vector α and passes through the point α. The spectrum vertex α is then also said to be *tame*.

Lemma 8.3. *Let $\alpha \in \mathfrak{M}$ be a tame vertex and $\Lambda \subset \mathbb{R}$ be a linear subspace containing α. Let there exist a vector $k \in \mathfrak{M} \setminus \Lambda$ such that $\langle \alpha, k \rangle > 0$. Then there exists a vector $\beta \in \mathfrak{M} \setminus \Lambda$ adjoint to α such that $\langle \beta, \alpha \rangle > 0$.*

Proof. We consider an orthogonal basis $e_1, \dots, e_n$ such that $e_1 = \alpha_1$ and $e_2, \dots, e_l \in \Lambda$, where $l = \dim \Lambda$. Because the vertex α is tame, the vector α is maximal among the vectors in $\mathfrak{M}$ with respect to the corresponding ordering $\prec$. Let β be the maximal vector in $k \in \mathfrak{M} \setminus \Lambda$.

In accordance with Lemma 8.2, the vector β is adjoint to α. Because $\beta \succcurlyeq k$, it follows that $\langle \alpha, \beta \rangle > 0$.

Lemma 8.4. *Let $\mathfrak{M} = -\mathfrak{M}$, let $\Lambda \subset \mathbb{R}^n$ be the linear space spanned by several vertices of the polyhedron $\mathfrak{C}$, and let the other vertices lie in $\Lambda^\perp$. Let there exist a vector $k \in \mathfrak{M} \setminus \Lambda$ that is not orthogonal to Λ. Then there exist a vertex $\alpha \in \Lambda$ of the spectrum and its adjoint vector β such that*

$$\beta \in \mathfrak{M} \setminus \Lambda, \qquad \langle \alpha, \beta \rangle > 0. \tag{8.2}$$

Proof. We consider the polyhedron $\operatorname{conv}(\mathfrak{M} \cap \Lambda)$. Its dimension (that of a manifold with boundary) is d, where $d = \dim \Lambda$ (otherwise the spectrum separates).

Let the vectors $h_1, \dots, h_J$ be the perpendiculars extending from the origin to its $(d-1)$-dimensional faces $f_1, \dots, f_J$ (or their extensions). We note that the base of at least one of the perpendiculars h_j is strictly inside the corresponding face. Indeed, the shortest perpendicular has this property.

It can be assumed that the above number j is 1. Let $\alpha_1, \dots, \alpha_N$ be the vertices in $\operatorname{conv}(\mathfrak{M} \cap \Lambda)$ lying in the face f_1. Because the vectors $\alpha_1, \dots, \alpha_N$ generate Λ, there exists $i \in \{1, \dots, N\}$ such that $\langle \alpha_i, k \rangle \neq 0$. Replacing k with $-k$ if necessary, we have $\langle \alpha_i, k \rangle > 0$. We assume that $i = 1$ in what follows.

The vector α_1 is maximal in $\mathfrak{M}$ with respect to the ordering $\prec$ corresponding to some orthogonal basis $e_1, \dots, e_n$ with $e_1 = h_1$ and $e_1, \dots, e_d \in \Lambda$.

We set $\beta = \max_{\xi \in \mathfrak{M}, \xi \notin \Lambda} \xi$, where the maximum is taken with respect to $\prec$. Because $\beta \succcurlyeq k$ and $\langle \alpha_1, k \rangle > 0$, we have $\langle \alpha_1, \beta \rangle > 0$.

The vector β is adjoint to $\alpha = \alpha_1$; this is proved the same as in Lemma 8.2.

9. The generation lemma

In what follows, for any linear subspace $\Lambda \subset \mathbb{R}^n$, we let $g(\Lambda) \subset SO(n)$ denote the subgroup of all orientation-preserving orthogonal transformations with unit determinant that coincide with the identity on the vectors in $\Lambda^\perp$, the orthogonal complement to Λ.

Lemma 9.1. *Let $\Lambda_1, \Lambda_2 \subset \mathbb{R}^n$ be two not mutually orthogonal linear subspaces of dimension not less than two. Then the groups $g(\Lambda_1)$ and $g(\Lambda_2)$ generate $g(\Lambda_1 \oplus \Lambda_2)$.*

Proof. The spaces Λ_1 and Λ_2 contain the subspaces $L_1 \subset \Lambda_1$ and $L_2 \subset \Lambda_2$ such that L_1 and L_2 are two-dimensional and are not mutually orthogonal. Then $g(L_1)$ and $g(L_2)$ generate $g(L_1 \oplus L_2)$ because of the following proposition.

Proposition 9.1. *Lemma* 9.1 *holds in the case where* $\dim \Lambda_1 = \dim \Lambda_2 = 2$.

There exist two sequences of subspaces $\Pi_0, \dots, \Pi_s$ and $\pi_1, \dots, \pi_s$ such that the following conditions are satisfied:

1. $L_1 \oplus L_2 = \Pi_0 \subset \Pi_1 \subset \dots \subset \Pi_s = \Lambda_1 \oplus \Lambda_2$;

2. $\dim \Pi_{j+1} = \dim \Pi_j + 1$, $j = 0, \dots, s-1$;
3. $\Pi_{j+1} = \Pi_j \oplus \pi_{j+1}$, $j = 0, \dots, s-1$;
4. $\dim \pi_j = 2$, $j = 1, \dots, s$; and
5. for any $j \in \{1, \dots, s\}$, the space π_j is contained in either Λ_1 or Λ_2.

We suppose that the following proposition holds.

Proposition 9.2. *Lemma* 9.1 *holds in the case where* $\dim \Lambda_2 = 2$ *and* $\dim \Lambda_1 \oplus \Lambda_2 = \dim \Lambda_1 + 1$.

Lemma 9.1 is then proved by consecutive application of Proposition 9.2, where we set $\Lambda_1 = \Pi_j$ and $\Lambda_2 = \pi_{j+1}$, $j = 0, \dots, s-1$.

It therefore remains to prove Propositions 9.1 and 9.2. The main tool used to prove these propositions is the Rashevsky–Zhou theorem [10]. In particular, this theorem states the following.

Let G_1 and G_2 be two connected subgroups in a finite-dimensional Lie group G, and let $\mathfrak{g}_1$, $\mathfrak{g}_2$, and $\mathfrak{g}$ be the corresponding Lie algebras. Let the minimal subalgebra containing $\mathfrak{g}_1$ and $\mathfrak{g}_2$ coincide with $\tilde{\mathfrak{g}}$. Then G_1 and G_2 generate the connected subgroup $\widetilde{G} \subset G$ whose Lie algebra is $\tilde{\mathfrak{g}}$.

In accordance with this theorem, it suffices to verify that the Lie algebras $\mathfrak{g}(\Lambda_1)$ and $\mathfrak{g}(\Lambda_2)$ of the groups $g(\Lambda_1)$ and $g(\Lambda_2)$ generate the Lie algebra $\mathfrak{g}(\Lambda_1 \oplus \Lambda_2)$ of the group $g(\Lambda_1 \oplus \Lambda_2)$.

Proof of Proposition 9.1. We have two cases:

a. $\Lambda_1 \cap \Lambda_2 = 0$ and

b. $\Lambda_1 \cap \Lambda_2 \neq 0$.

Case b follows if we set $\dim \Lambda_1 = 2$ in Proposition 9.2, and we therefore restrict ourself to case a. Here, we can assume that the ambient space dimension is $n = 4$ and that Λ_1 is spanned by the first two basis vectors of an orthonormal basis. Then $\mathfrak{g}(\Lambda_1) \subset so(4)$ is a one-dimensional subspace generated by the matrix

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9.1}$$

Let the column vectors $u, v \in \mathbb{R}^4$ constitute a basis in Λ_2. Then $\mathfrak{g}(\Lambda_2) \subset so(4)$ is the one-dimensional subspace generated by the matrix

$$A_2 = uv^{\mathrm{T}} - vu^{\mathrm{T}} = \begin{pmatrix} 0 & a & c & d \\ -a & 0 & e & f \\ -c & -e & 0 & b \\ -d & -f & -b & 0 \end{pmatrix}. \tag{9.2}$$

Because $\Lambda_1 \oplus \Lambda_2 = \mathbb{R}^4$, we have $b \neq 0$. Because Λ_1 and Λ_2 are not orthogonal to each other, at least one of the coefficients c, d, e, or f is nonzero. It remains to use the following proposition.

Proposition 9.3. *Let a, b, c, d, e, and f be real numbers such that*

$$b \neq 0, \qquad c^2 + d^2 + e^2 + f^2 \neq 0.$$

Then the minimal Lie algebra containing matrices (9.1) *and* (9.2) *is* $so(4)$.

Proposition 9.3 is verified by direct calculation.

Proof of Proposition 9.2. It can be assumed that the dimension of the ambient space is $n = \dim \Lambda_1 + 1$ and that Λ_1 is spanned by the first $n-1$ basis vectors of an orthonormal basis. Then $\mathfrak{g}(\Lambda_1)$ consists of all skew-symmetric matrices with a zero last column and a zero last row.

Let $u, v \in \mathbb{R}^n$ be a basis in Λ_2. The algebra $\mathfrak{g}(\Lambda_2)$ is one-dimensional and is generated by the matrix

$$A = uv^{\mathrm{T}} - vu^{\mathrm{T}} = \begin{pmatrix} * & w \\ w^{\mathrm{T}} & 0 \end{pmatrix}, \quad 0 \neq w \in \mathbb{R}^{n-1}.$$

It remains to use following proposition.

Proposition 9.4. *Let $L \subset \mathbb{R}^n$ be a linear subspace spanned by the first $n-1$ basis vectors. Then the minimal Lie algebra containing the Lie algebra of the group $g(L)$ and the matrix A is $so(n)$.*

Proposition 9.4 is verified by direct calculation.

10. Completing the proof of Theorem 2.1

We consider the polyhedron $\mathfrak{C}$. Its positive edges are split into several equivalence classes $\mathbf{I}_1, \dots, \mathbf{I}_s$. Two edges are in the same class if and only if passing from one of them to the other along positive edges is possible. If there are no positive edges, we assume that no classes $\mathbf{I}_j$ exist.

We consider some class $\mathbf{I}_j$. A vertex α and the adjoint vector β such that $\langle \alpha, \beta \rangle > 0$ (Lemma 8.1) corresponds to each edge in this class.

To each class $\mathbf{I}_j$, there corresponds $L_j \subset \mathbb{R}^n$, the minimal linear subspace containing the edges from $\mathbf{I}_j$. The corresponding groups $g(\alpha, \beta)$ generate $g(L_j)$ (Lemma 9.1).

We note that the linear space L_0 generated by the vectors $\alpha_1, \dots, \alpha_K$ directed from the origin to the vertices of $\mathfrak{C}$ not in $L_1 \cup \dots \cup L_s$ is such that

$$L_1 \oplus \dots \oplus L_s \oplus L_0 = \mathbb{R}^n.$$

If this is not so, then $\mathfrak{M}$ is in a positive-codimension subspace of $\mathbb{R}^n$, i.e., the spectrum separates.

Any of the vertices α_j is tame (otherwise a positive edge would emanate from it). We consider a fixed $j \in \{1, \dots, K\}$. If we have $\langle \alpha_j, k \rangle = 0$ for any $k \in \mathfrak{M}$, then the spectrum separates. Therefore, we can assume that there exists a vector $k_0 \in \mathfrak{M}$ such that $\langle \alpha_j, k_0 \rangle > 0$ (we here use the symmetry of the spectrum). In accordance with Lemma 8.3, there exists a vector β_j adjoint to α_j such that $\langle \alpha_j, \beta_j \rangle > 0$.

Let l_j be the two-dimensional plane spanned by α_j and β_j, and let $g_j = g(\alpha_j, \beta_j) \equiv g(l_j)$ be the corresponding subgroup in $SO(n)$. Then

$$\mathbb{R}^n = L_1 \oplus \dots \oplus L_s \oplus l_1 \oplus \dots \oplus l_K,$$

and the symbol $F_{(M)}$ remains invariant under the action of the groups $g(L_1), \dots, g(L_s)$ and $g(l_1), \dots, g(l_K)$.

Let Λ_1 and Λ_2 be two spaces from the set $L_1, \dots, L_s, l_1, \dots, l_K$. If Λ_1 and Λ_2 are not orthogonal, then the corresponding groups $g(\Lambda_1)$ and $g(\Lambda_2)$ generate $g(\Lambda_1 \oplus \Lambda_2)$. Then the spaces Λ_1 and Λ_2 can be replaced with $\Lambda_1 \oplus \Lambda_2$.

Uniting the spaces thus (as long as this is possible), we eventually obtain the decomposition of $\mathbb{R}^n$

$$\mathbb{R}^n = Q_1 \oplus \dots \oplus Q_N,$$

where

a. the subspaces Q_j are pairwise orthogonal,

b. each subspace Q_j is generated by vertices of the polyhedron $\mathfrak{C}$, and

c. the groups $g(Q_j)$ preserve the symbol $F_{(M)}$.

We now use Lemma 8.4. We take Q_1 as Λ. The vector k in Lemma 8.4 exists because otherwise the spectrum separates. Conditions (8.2) imply that the vector β is not orthogonal to some Q_j, $j \neq 1$. Let $Q_{j_1}, \dots, Q_{j_s}$ be spaces not orthogonal to β. Then the groups $g(Q_1), g(Q_{j_1}), \dots, g(Q_{j_s})$ and $g(\alpha, \beta)$ generate $g(Q_1 \oplus Q_{j_1} \oplus \dots \oplus Q_{j_l})$. We thus reduce the number of terms in the decomposition of $\mathbb{R}^n$. Proceeding by induction, we then find that the symbol $F_{(M)}$ is invariant under the action of $SO(n)$.

We have thus proved that for a nonseparating spectrum, $F_{(M)} = c_M \langle p, p\rangle^{M/2}$, $c_M = \text{const}$. But then the operator

$$\widehat{\Phi} = \widehat{F} - c_M \widehat{H}^{M/2}$$

is polynomial in the derivatives, commutes with $\widehat{H}$, and the degree of $\widehat{\Phi}$ is less than M. We then repeat the argument with $\widehat{F}$ replaced with $\widehat{\Phi}$, and so on. The proof of Theorem 2.1 is finished.

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