Doklady Mathematics, Vol. 70, No. 2, 2004, pp. 807–810. Translated from Doklady Akademii Nauk, Vol. 398, No. 3, 2004, pp. 314–318.

MATHEMATICAL PHYSICS

Conservation Laws in Quantum Systems on a Torus

Academician V. V. Kozlov and Corresponding Member of the RAS D. V. Treshchev

Received May 17, 2004

1. INTRODUCTION

Let

$$H = \frac{1}{2}(p, p) + \varepsilon V(\varepsilon) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \varepsilon V(x_1, x_2, \ldots, x_n) \quad (1)$$

be a classical Hamiltonian, $x \bmod 2\pi$ be generalized angular coordinates, p be conjugate momenta, $V: \mathbb{T}^n \to \mathbb{R}$ be the potential energy, and ε be a small parameter. By the quantization operation

$$x \mapsto \hat{x}, \quad p \mapsto \hat{p} = -i\hbar\frac{\partial}{\partial x},$$

the Hamiltonian is put in correspondence with the second-order differential operator

$$\hat{H} = -\frac{\hbar^2\Delta}{2} + \varepsilon\hat{V}(x), \quad (2)$$

where Δ is a Laplace operator and $\hat{x}$ and $\hat{V}(x)$ are the operators of multiplication by x and $V(x)$, respectively.

Poincaré studied the problem of whether a classical system with Hamiltonian (1) has an additional integral represented by a series in powers of the small parameter,

$$F = F_0(x, y) + \varepsilon F_1(x, y) + \varepsilon^2 F_2(x, y) + \ldots, \quad (3)$$

where the coefficients F_k are smooth 2π-periodic functions of angular variables $x_1, x_2, \ldots, x_n$. Poincaré called such integrals single-valued. Of course, it is assumed that functions (1) and (3) are independent of each other. An important role in the existence conditions for new single-valued integrals is played by the structure of the resonance set

$$\bigcup_{k \in Z^n} \{p \in \mathbb{R}^n: (k, p) = 0, v_k \neq 0\}, \quad (4)$$

where v_k are the expansion coefficients of the potential energy expanded in a Fourier series:

$$V(x) = \sum v_k e^{i(k, x)}$$

(see [1, 2] for more detail).

We consider a similar problem, namely, the existence of single-valued operators

$$\hat{A} = \hat{A}_0 + \varepsilon\hat{A}_1 + \ldots \quad (5)$$

commuting with Hamiltonian operator (2): $[\hat{H}, \hat{A}] = 0$. Here, $[\cdot, \cdot]$ denotes the quantum commutator

$$[\hat{A}, \hat{B}] = -\frac{1}{i\hbar}(\hat{A} \circ \hat{B} - \hat{B} \circ \hat{A}).$$

Single-valuedness means that the coefficients of formal power series (5) in ε have the form

$$\hat{A}_k = \sum_{\mu} a_{\mu}^{(k)}(x)(-i\hbar\partial)^{\mu}, \quad \mu \in \mathbb{Z}_+^n, \quad (6)$$

where $\partial^{\mu} = \partial_n^{\mu_1}\partial_n^{\mu_2} \ldots \partial_n^{\mu_n}$, $\partial_j = \frac{\partial}{\partial x_i}$, and $a_{\mu}^{(k)}(x)$ are smooth 2π-periodic (generally complex-valued) functions of $x_1, x_2, \ldots, x_n$. Assume that operators (5) and (2) are independent of each other. This assumption will be refined below.

To the differential operator (6), we assign its symbol

$$A_k(x, p) = \mathrm{symb}\hat{A}_k = \sum_{\mu} a_{\mu}^{(k)}(x)p^{\mu},$$

i.e., a formal power series in $p = (p_1, p_2, \ldots, p_n)$. Since all derivatives in (6) appear on the right-hand side, there is a one-to-one correspondence between the operators and their symbols.

Below we will need the following two properties of symbols:

$$\mathrm{symb}[\hat{A}, \hat{V}] = -\frac{1}{i\hbar}(A(x, p - i\hbar\partial) - A(x, p))V(x), \quad (7)$$

$$\mathrm{symb}[\hat{A}, -\Delta] = (2(p, \partial) - i\hbar\partial^2)A(x, p). \quad (8)$$

Here, $\partial = (\partial_1, \partial_2, \ldots, \partial_n)$ and the Laplace operator $\partial_2 = (\partial, \partial) = \Delta$ both act on functions of $x_1, x_2, \ldots, x_n$. Relations (7) and (8) can be checked by direct calculation.

Moscow State University, Leninskie gory, Moscow, 119992 Russia
e-mail: dtresch@mech.math.msu.su

2. RESONANCES

Expanding the left-hand side of $[\hat{A}, \hat{H}] = 0$ into series in powers of ε, we obtain a chain of commutation relations:

$$[\hat{A}_0, -\Delta] = 0, \tag{9}$$

$$[\hat{A}_0, \hat{V}] + \frac{1}{2}[\hat{A}_1, -\Delta] = 0, \tag{10}$$

$$\dots\dots\dots\dots\dots\dots$$

First, we show that $\hat{A}_0$ does not depend on x. For this purpose, we use (9) and (8):

$$(2(p, \partial) - i\hbar\partial^2)A_0(x, p) = 0. \tag{11}$$

The function A_0 is represented as a Fourier series

$$A_0 = \sum a_k(p)e^{i(k,x)}, \quad k \in \mathbb{Z}^n.$$

Then (11) yields an infinite series of relations

$$[2(p, k) + \hbar(k, k)]a_k(p) = 0, \quad k \in \mathbb{Z}^n.$$

When $k \neq 0$, the expression in square brackets vanishes only in one hyperplane on $\mathbb{R}^n = \{p\}$. Therefore, $a_k(p) \equiv 0$ for all $k \neq 0$, as required.

This fact is similar to Poincaré's remark that F_0 in (3) depends only on the canonical momenta. If $\hat{A}_0$ does not depend on x, then (9) is obviously fulfilled.

Now we examine relation (10). Using identities (7) and (8) and Fourier expansions, we obtain

$$-\frac{1}{i\hbar}v_k(A_0(p - i\hbar\partial) - A_0(p))e^{i(k,x)}$$

$$+\frac{1}{2}a_k(p)(2i(p, k) + i\hbar(k, k))e^{i(k,x)} = 0.$$

Here, $a_k(p)$ are the Fourier coefficients of $A_1(x, p)$. These equalities can be represented in an equivalent form:

$$\frac{1}{\hbar}(A_0(p + \hbar k) - A_0(p))e^{i(k,x)}v_k$$
$$+\frac{1}{2\hbar}((p + \hbar k, k + \hbar k) - (p, p))a_k(p) = 0 \tag{12}$$

for all $k \in \mathbb{Z}^n$.

Now we set

$$2(p, k) + \hbar(k, k) = 0, \quad v_k \neq 0. \tag{13}$$

Then (12) implies that

$$A_0(p + \hbar k) = A_0(p). \tag{14}$$

Thus, the role of resonance set (4) in quantum mechanics is played by the union of hyperplanes (13). In contrast to the classical case, these hyperplanes do not pass through the origin $p = 0$ and, if they are infinitely many, they are accumulated in the neighborhood of "points at infinity."

Our goal is to prove that, in the generic case, (14) implies that A_0 is a function of (p, p). If $A_0 = f((p, p))$, then $\hat{A}_0 = f(\Delta)$. This implies (at least, at the formal level) that operator (5) can be expressed in terms of Hamiltonian (2) and, consequently, does not generate new conservation laws. Indeed, the operator $\hat{A} - f(-2\hat{H})$ commutes with $\hat{H}$, and its expansion in powers of ε does not contain a free term:

$$\hat{A} - f(-2\hat{H}) = \varepsilon(\hat{A}_0' + \varepsilon\hat{A}_1' + \dots).$$

The operator in parentheses on the right-hand side commutes with $\hat{H}$, and $\hat{A}_0'$ does not depend on $x_1, x_2, \dots, x_n$ and is a function of the Laplace operator. Hence,

$$\hat{A} = f(-2\hat{H}) + \varepsilon g(-2\hat{H}) + \dots.$$

The genericity of the situation lies in the fact that the Fourier expansion of the potential energy contains virtually all harmonics. This suggests that the classical resonance set (4) is dense everywhere in $\mathbb{R}^n = \{p\}$. In turn, this immediately implies that $F_0(p)$ and (p, p) are dependent on each other.

As mentioned above, the situation in the quantum case is more complicated. For example, when $n = 1$, it follows from (13) that $p = -\hbar k/2$. Consequently, (14) takes the form

$$A_0(\hbar k/2) = A_0(-\hbar k/2)$$

for all integers k (of course, if all $v_k \neq 0$). We want to show that A_0 is an even function of p (or, equivalently, A_0 is a function of p^2). This is true if A_0 is a polynomial in p. However, in the general case, $A_0(p)$ is not necessarily an even function. A simple example is $A_0 = \sin\frac{2\pi p}{\hbar}$.

Remark. Letting $\hbar$ tend to zero in (13), we obtain a resonance relation in the classical case. This corresponds to the well-known transition from quantum to classical mechanics as the Dirac constant approaches zero.

The possibility of extending Poincaré's theory to quantum systems was discussed in [3, 4]. Specifically, the problem of small perturbations of a linear oscillatory system (the Friedrichs model) was considered. However, this problem is not directly related to Poincaré's theory, since the unperturbed system is strongly degenerate and the resonance relation differs from (13). To bring the exposition of [3, 4] to Poincaré's theory, we should consider "quasilinear" Hamiltonian systems that depend nontrivially on another parameter μ (in addition to ε). In that case, there are several resonances (at different values of μ), which prevent the existence of a commuting operator depending analytically on ε and μ. For classical systems, this idea was implemented in [5] (see also [2]).

3. NONINTEGRABILITY CONDITIONS

Our problem is reduced to the following one: find the conditions under which (13) and (14) imply that $A_0(p)$ is a function of (p, p). First, consider the case where $\hat{A}$ is a polynomial in $\partial_1, \partial_2, \ldots, \partial_n$ with single-valued coefficients on $\mathbb{T}^n$. Then A_0 is a polynomial in $p_1, p_2, \ldots, p_n$. Note that, in all known problems, operators commuting with the Hamiltonian operator are polynomials (or functions of polynomials).

The spectrum S of $V(x)$ is the set of integer vectors $k \in \mathbb{Z}^n$ such that $v_k \neq 0$. Since V is real-valued, the set S is invariant under the reflection $k \mapsto -k$. The set $S \subset \mathbb{C}^n$ is called a key set (or a uniqueness set) for the space of polynomials on $\mathbb{C}^n$ if any polynomial vanishing at points of S vanishes identically on $\mathbb{C}^n$.

Clearly, $\mathbb{Z}^n$ is a key set. Let us give a sufficient condition for S to be a key set. A straight line in $\mathbb{C}^n$ that contains an infinite number of distinct points of S and passes through the origin is called a support line. If the set of all support lines forms a key set, then S is also a key set. For $n = 2$, this condition means that there are an infinite number of distinct support lines.

Theorem 1. *If the spectrum S of a quantum system is a key set for the space of polynomials in $\mathbb{C}^n$, then this system does not admit operators of form* (5) *that are polynomials in derivations and are independent of Hamiltonian operator* (2).

Proof. We prove this theorem in the simplest nontrivial case of $n = 2$. Let $p = (p_1, p_2)$ and $k = (k_1, k_2)$. The resonance relation (13) is fulfilled if we set

$$p_1 = -\frac{\hbar k_1}{2} + \alpha k_2, \quad p_2 = -\frac{\hbar k_2}{2} - \alpha k_1,$$

where α is a complex parameter. Hence, (14) takes the form

$$A_0\left(-\frac{\hbar k_1}{2} + \alpha k_2, -\frac{\hbar k_2}{2} - \alpha k_1\right) = A_0\left(\frac{\hbar k_1}{2} + \alpha k_2, \frac{\hbar k_2}{2} - \alpha k_1\right). \tag{15}$$

Here, we have set $A(\cdot) = A_0(-i(\cdot))$.

First, we note that $p \mapsto A_0(p)$ is an even function. Indeed, the polynomial

$$A_0(p) - A_0(-p) \tag{16}$$

vanishes at points of $\frac{S}{2}$. Since S is a key set for the space of polynomials over $\mathbb{C}$, difference (16) vanishes identically.

Furthermore, we differentiate (15) with respect to α and then set $\alpha = 0$:

$$\left(\frac{\partial A_0}{\partial p_1}\bigg|_{-\hbar k/2} - \frac{\partial A_0}{\partial p_1}\bigg|_{\hbar k/2}\right)k_2 - \left(\frac{\partial A_0}{\partial p_2}\bigg|_{-\hbar k/2} - \frac{\partial A_0}{\partial p_2}\bigg|_{\hbar k/2}\right)k_1 = 0.$$

Since $A_0(p)$ is an even function, this equality takes the form

$$\frac{\partial A_0}{\partial p_1}p_2 - \frac{\partial A_0}{\partial p_2}p_1 = 0 \tag{17}$$

at points of $\frac{S}{2}$. Since S is a key set, the polynomial on the left-hand side of (17) vanishes identically. However, (17) then implies that A_0 is a function of (p, p), as required.

Theorem 1 can easily be extended to the more general case where $A_0(p_1, \ldots, p_n)$ is a meromorphic function in the compactified space $\overline{\mathbb{C}}^n$ of n complex variables. In that case (by the Weierstrass–Hurwitz theorem), A_0 is a rational function of $p_1, \ldots, p_n$ (the ratio of two polynomials).

4. POLYNOMIAL OPERATORS

Now we set $\varepsilon = 1$ in Hamiltonian operator (2) and consider the problem of the existence of a differential operator

$$\hat{A} = \sum a_\mu(x)(-i\hbar\partial)^\mu, \quad \mu_1 + \mu_2 + \ldots + \mu_n \le M, \tag{18}$$

commuting with the Hamiltonian operator. Here, the coefficients are periodic functions of $x_1, x_2, \ldots, x_n$. Operator (18) is a polynomial in $\partial_1, \ldots, \partial_n$; and M is its degree. This operator can be represented as a sum of homogeneous forms (with respect to ∂_j):

$$\hat{A}_M + \hat{A}_{M-1} + \ldots + \hat{A}_0. \tag{19}$$

Here, k is the degree of the homogeneous operator $\hat{A}_k$. Let

$$A(x, p) = A_M(x, p)A_{M-1}(x, p) + \ldots + A_0(x, p) \tag{20}$$

be the symbol of operator (18), (19), and let $A_k(x, p)$ be a polynomial in $p_1, p_2, \ldots, p_n$ of degree k with single-valued coefficients on $\mathbb{T}^n$.

The condition $[\hat{H}, \hat{A}] = 0$ is reduced to

$$2\left(p, \frac{\partial}{\partial x}\right)A(x, p) - i\hbar\Delta A(x, p) - \frac{1}{i\hbar}(A(x, p - i\hbar\partial) - A(x, p))V(x) = 0. \tag{21}$$

Substituting (20) into this equality and setting the homogeneous forms on p equal to zero, we obtain a chain of equations for sequentially determining A_M, $A_{M-1}, \ldots, A_0$.

Setting the terms to the power $M + 1$ in (21) equal to zero, we obtain the equation

$$\left(p, \frac{\partial}{\partial x}\right) A_M(x, p) = 0,$$

which implies that the leading form A_M of polynomial (20) does not depend on x.

Setting the sum of the terms to the power M on the left-hand side of (21) equal to zero gives

$$2\left(p, \frac{\partial}{\partial x}\right) A_{M-1} - i\hbar \Delta A_M = 0.$$

Since A_M depends only on p, A_{M-1} does not depend on x.

Remark. The last property has a natural analogue in the case of classical Hamiltonian equations with Hamiltonian (1) (where $\varepsilon = 1$): the sums of homogeneous forms of even and odd powers of a first integral that is a polynomial in momenta are also first integrals.

Taking into account these properties and setting the sum of the terms to the power $M - 2$ equal to zero gives

$$2\left(p, \frac{\partial}{\partial x}\right) A_{M-2} + \left(\frac{\partial A_M}{\partial p}, \frac{\partial}{\partial x}\right) V = 0.$$

This equation is well known in the integrability theory of classical Hamiltonian systems (see [1, 2]). In particular, this equation implies that the functions A_M and (p, p) are dependent at points of the usual resonance set $\{(p, k) = 0, v_k \neq 0, k \neq 0\}$. The following result can easily be proven by using this fact.

Theorem 2. *Assume that resonance set* (4) *is a key set for the space of polynomials in n variables.*

Then any operator of form (18) *commuting with the Hamiltonian operator* $\hat{H}$ *is a polynomial in* $\hat{H}$ *with constant coefficients.*

In particular, if the expansion of $V(x)$ contains all harmonics, then the quantum system does not admit nontrivial differential polynomial operators commuting with the Hamiltonian operator.

In the case of two degrees of freedoms, the condition of Theorem 2 is obviously fulfilled if the points of S lie on an infinite number of distinct lines passing through the origin.

We say that S is separable if its points lie on the union of two subspaces of dimension less than $<n$ that intersect orthogonally at the origin. If the spectrum is separable, then Hamiltonian operator (2) can be decomposed into the sum of the Hamiltonian operators of two subsystems of the original quantum system with n degrees of freedoms. Each of these two partial Hamiltonian operators is independent of $\hat{H}$ and commutes with $\hat{H}$.

Theorem 3. *Assume that the potential energy V is a trigonometric polynomial* (*the spectrum S is finite*).

Then a nontrivial polynomial operator commuting with the Hamiltonian operator exists if and only if the spectrum of the quantum system is separable.

This theorem is more complicated than Theorem 2. Its proof is based on a careful analysis of the full equation (21).

We hypothesize that Theorem 3 is valid in the general case of an infinite S. For classical Hamiltonian systems, this hypothesis was discussed in [6–8]. Unfortunately, it has not yet been completely proven even for systems with two degrees of freedom. It was shown in [8] that, if such a system admits an irreducible polynomial integral of degree n and its spectrum lies on n straight lines passing through the origin, then these lines are separated by an angle of $\frac{\pi}{n}$. This result indicates deep relationships between continuous and discrete symmetries of dynamic equations.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (project nos. 02-01-00400, 02-01-01059), by a grant of the President of the Russian Federation (MD-261.2003.01), and by INTAS (project no. 00-221).

REFERENCES

1. H. Poincaré, *Selected Works* (Nauka, Moscow, 1972), Vol. 1 [in Russian].
2. V. V. Kozlov, *Symmetry, Topology, and Resonances in Hamiltonian Mechanics* (Udmurt. Gos. Univ., Izhevsk, 1995) [in Russian].
3. I. E. Antoniou and I. Prigogine, Phys. A (Amsterdam) **192**, 443–464 (1993).
4. I. E. Antoniou and S. Tusaki, Int. J. Quant. Chem. **46**, 425–474 (1993).
5. V. V. Kozlov, Vestn. Mosk. Gos. Univ., Ser. 1: Mat. Mekh., No. 1, 110–115 (1976).
6. V. V. Kozlov and D. V. Treshchev, Mat. Sb. **135** (177), 119–138 (1988).
7. M. L. Byalyi, Funkts. Anal. Ego Prilozh. **21** (4), 64–65 (1987).
8. N. V. Denisova and V. V. Kozlov, Mat. Sb. **191** (2), 43–63 (2000).