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On the Fall of a Heavy Rigid Body in an Ideal Fluid

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Abstract—We consider a problem about the motion of a heavy rigid body in an unbounded volume of an ideal irrotational incompressible fluid. This problem generalizes a classical Kirchhoff problem describing the inertial motion of a rigid body in a fluid. We study different special statements of the problem: the plane motion and the motion of an axially symmetric body. In the general case of motion of a rigid body, we study the stability of partial solutions and point out limiting behaviors of the motion when the time increases infinitely. Using numerical computations on the plane of initial conditions, we construct domains corresponding to different types of the asymptotic behavior. We establish the fractal nature of the boundary separating these domains.

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1. EQUATIONS OF MOTION AND SPECIAL CASES

Let us consider the motion of a rigid body in a homogeneous gravity field in an infinite volume of irrotational incompressible fluid resting at infinity. First let us give general equations of motion of a body in a fluid under the action of an external force field:

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{p} \times \frac{\partial H}{\partial \boldsymbol{p}} + \boldsymbol{K}, \qquad \dot{\boldsymbol{p}} = \boldsymbol{p} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{F}, \tag{1.1}$$

where $\boldsymbol{F}$ and $\boldsymbol{K}$ are the total force and moment applied to the body. These equations go back to Kirchhoff. If the external forces have the potential nature, then Eqs. (1.1) supplemented by the equations for directional cosines and for coordinates of a fixed point in the body, can be written as follows:

$$\begin{gathered}
\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{p} \times \frac{\partial H}{\partial \boldsymbol{p}} + \boldsymbol{\alpha} \times \frac{\partial H}{\partial \boldsymbol{\alpha}} + \boldsymbol{\beta} \times \frac{\partial H}{\partial \boldsymbol{\beta}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \\
\dot{\boldsymbol{p}} = \boldsymbol{p} \times \frac{\partial H}{\partial \boldsymbol{M}} - \frac{\partial H}{\partial x_1}\boldsymbol{\alpha} - \frac{\partial H}{\partial x_2}\boldsymbol{\beta} - \frac{\partial H}{\partial x_3}\boldsymbol{\gamma}, \\
\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \frac{\partial H}{\partial \boldsymbol{M}}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \frac{\partial H}{\partial \boldsymbol{M}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{M}}, \\
\dot{x}_1 = \left(\boldsymbol{\alpha}, \frac{\partial H}{\partial \boldsymbol{p}}\right), \quad \dot{x}_2 = \left(\boldsymbol{\beta}, \frac{\partial H}{\partial \boldsymbol{p}}\right), \quad \dot{x}_3 = \left(\boldsymbol{\gamma}, \frac{\partial H}{\partial \boldsymbol{p}}\right),
\end{gathered} \tag{1.2}$$

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where vectors $\boldsymbol{p}$, $\boldsymbol{M}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are the projections of linear momentum, angular momentum, and unit vectors along axes in the fixed frame of reference on the axes connected with the body; and x_1, x_2, x_3 are the projections of position vector of the origin of moving coordinate system on the fixed axes. The Hamiltonian of system (1.2) has the form

$$\begin{gathered} H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\mathbf{B}\boldsymbol{M}, \boldsymbol{p}) + \frac{1}{2}(\mathbf{C}\boldsymbol{p}, \boldsymbol{p}) + U, \\ U = \mu\big(x_3 + (\boldsymbol{r}, \boldsymbol{\gamma})\big), \quad \mu = \mu_{\mathrm{b}} - \mu_{\mathrm{f}}, \quad \boldsymbol{r} = \frac{\mu_{\mathrm{b}}\boldsymbol{r}_{\mathrm{b}} - \mu_{\mathrm{f}}\boldsymbol{r}_{\mathrm{f}}}{\mu_{\mathrm{b}} - \mu_{\mathrm{f}}}; \end{gathered} \tag{1.3}$$

here, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are symmetric matrices determined by geometry of the body and by its inertial properties, μ_{b}, μ_{f} are the weight of the body and the weight of the displaced fluid, and $\boldsymbol{r}_{\mathrm{b}}$, $\boldsymbol{r}_{\mathrm{f}}$ are the position vectors of the center of gravity and the center of pressure in moving axes. The case $\mu_{\mathrm{b}} = \mu_{\mathrm{f}}$ (a suspended body) will be also studied below.

By straightforward calculations we can verify that there are three integrals of motion (one of which explicitly contains the time):

$$(\boldsymbol{p}, \boldsymbol{\alpha}) = P_1, \quad (\boldsymbol{p}, \boldsymbol{\beta}) = P_2, \quad (\boldsymbol{p}, \boldsymbol{\gamma}) + \mu t = P_3.$$

This means that linear momentum of the body + fluid system can be represented in the form

$$\boldsymbol{p} = P_1\boldsymbol{\alpha} + P_2\boldsymbol{\beta} + (P_3 - \mu t)\boldsymbol{\gamma}, \tag{1.4}$$

i.e., vector $\boldsymbol{P} = (P_1, P_2, P_3)$ is the initial impulse (impact, according to Chaplygin) in the fixed frame of reference.

By the choice of zero time point (for $\mu_{\mathrm{b}} \neq \mu_{\mathrm{f}}$) and by rotation of the fixed axes we can obtain $P_2 = P_3 = 0$. In what follows, we consider this to be fulfilled.

Substituting (1.4) into equations of motion (1.2), we obtain a self-contained system with respect to $\boldsymbol{M}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, which can be written in the Hamiltonian form:

$$\begin{gathered} \dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial \bar{H}}{\partial \boldsymbol{M}} + \boldsymbol{\alpha} \times \frac{\partial \bar{H}}{\partial \boldsymbol{\alpha}} + \boldsymbol{\beta} \times \frac{\partial \bar{H}}{\partial \boldsymbol{\beta}} + \boldsymbol{\gamma} \times \frac{\partial \bar{H}}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \frac{\partial \bar{H}}{\partial \boldsymbol{M}}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \frac{\partial \bar{H}}{\partial \boldsymbol{M}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial \bar{H}}{\partial \boldsymbol{M}}, \end{gathered} \tag{1.5}$$

with the Hamiltonian explicitly depending on time:

$$\bar{H} = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\mathbf{B}\boldsymbol{M}, P_1\boldsymbol{\alpha} - \mu t\boldsymbol{\gamma}) + \frac{1}{2}\big(\mathbf{C}(P_1\boldsymbol{\alpha} - \mu t\boldsymbol{\gamma}), P_1\boldsymbol{\alpha} - \mu t\boldsymbol{\gamma}\big) + \mu(\boldsymbol{r}, \boldsymbol{\gamma}). \tag{1.6}$$

Remark. Equations (1.2) in various but equivalent forms can be found in papers by V.A. Steklov [17], D.N. Goryachev [5], and S.A. Chaplygin [20]. V.V. Kozlov [11], we believe, was the first who reduced them to an elegant nonautonomous form, using the representation (1.4) (in the form of Poincaré equations).

Let us point out some special cases when Eqs. (1.5) can be simplified. They are indicated in [11, 19].

1.1. Motion without an initial impulse [11]. Let the initial impulse be equal to zero: $P_1 = 0$. The equations of motion for $\boldsymbol{M}$, $\boldsymbol{\gamma}$ in a closed form represent a (nonautonomous) Hamiltonian system on $e(3)$ (see below) with the Hamiltonian

$$\bar{H} = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) - \mu t(\mathbf{B}\boldsymbol{M}, \boldsymbol{\gamma}) + \frac{1}{2}\mu^2 t^2(\mathbf{C}\boldsymbol{\gamma}, \boldsymbol{\gamma}) + \mu(\boldsymbol{r}, \boldsymbol{\gamma}).$$

If, in addition, the body has three planes of symmetry intersecting in the center of gravity, then the Hamiltonian can be simplified further: $\mathbf{B} = 0$, $\boldsymbol{r} = 0$.

1.2. Suspended body [19]. In [19], Chaplygin also indicated the case when the gravitation is balanced by the Archimedean force (an average density of a body is equal to the density of fluid, and, hence, $\mu_\mathrm{b} = \mu_\mathrm{f}$), however, the center of gravity of the body does not coincide with the center of gravity of the displaced fluid volume. Thus, the body is under the action of a pair of forces, and its total linear momentum in the fixed frame of reference is conserved, i.e.,

$$\boldsymbol{p} = P_1\boldsymbol{\alpha} + P_2\boldsymbol{\beta} + P_3\boldsymbol{\gamma},$$

where $\boldsymbol{P} = (P_1, P_2, P_3) = \mathrm{const}$. As above, by the choice of fixed axes we can obtain the equality $P_2 = 0$. Thus, in this case the evolution of vectors $\boldsymbol{M}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ can be described by an autonomous Hamiltonian system with the Hamiltonian function

$$\bar{H} = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\mathbf{B}\boldsymbol{M}, P_1\boldsymbol{\alpha} + P_3\boldsymbol{\gamma}) + \frac{1}{2}\big(\mathbf{C}(P_1\boldsymbol{\alpha} + P_3\boldsymbol{\gamma}), P_1\boldsymbol{\alpha} + P_3\boldsymbol{\gamma}\big) + \mu_\mathrm{b}(\boldsymbol{r}, \boldsymbol{\gamma}),$$

where $\boldsymbol{r}$ is the vector connecting the center of gravity of the body with the center of pressure.

If the initial impulse is directed along the vertical axis, $\boldsymbol{p} = P\boldsymbol{\gamma}$, then the evolution of vectors $\boldsymbol{M}$, $\boldsymbol{\gamma}$ ($\boldsymbol{\gamma}$ is directed along the field of gravity) is described by a system with Poisson's bracket which is determined by the algebra $e(3)$ (i.e., $\{M_i, M_j\} = \varepsilon_{ijk}M_k$, $\{M_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k$, $\{\gamma_i, \gamma_j\} = 0$), and by the Hamiltonian function

$$\bar{H} = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + P(\mathbf{B}\boldsymbol{M}, \boldsymbol{\gamma}) + \frac{1}{2}P^2(\mathbf{C}\boldsymbol{\gamma}, \boldsymbol{\gamma}) + \mu_\mathrm{b}(\boldsymbol{r}, \boldsymbol{\gamma}). \tag{1.7}$$

We have equations of motion in the explicit form

$$\dot{\boldsymbol{M}} = \frac{\partial H}{\partial \boldsymbol{M}} \times \boldsymbol{M} + \frac{\partial H}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{M}}.$$

In [19] Chaplygin indicated the case when Eqs. (1.7) are integrable with an additional integral of fourth degree in components of an angular momentum. The form of the integral is similar to the Kovalevskaya integral.

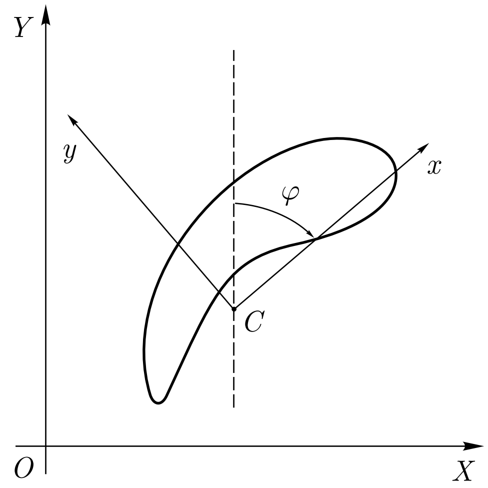


Fig. 1.

1.3. Plane-parallel motion [5, 9, 17, 20]. The plane-parallel motion of a rigid body is determined by the invariant relations $M_1 = M_2 = 0$, $\alpha_3 = \gamma_3 = 0$. It is easy to show that the dynamical symmetry of the body with respect to the considered (invariant) plane is a necessary condition for the existence of such motions. This leads to the relations

$$b_{11} = b_{22} = b_{33} = b_{12} = 0, \quad c_{13} = c_{23} = 0.$$

In addition, it can be shown that in this case, by choosing the axes connected with the body, one can obtain the equality $\mathbf{B} = 0$ and the diagonal matrix $\mathbf{C}$. Let an angle of rotation of the moving axes relative to the fixed axes be clockwise countered, as it is shown in Fig. 1, then for the unit vectors of fixed axes we have

$$\alpha_1 = \sin\varphi, \quad \alpha_2 = -\cos\varphi, \quad \gamma_1 = \cos\varphi, \quad \gamma_2 = \sin\varphi.$$

For an angle of rotation we obtain the nonautonomous second-order equation

$$a_3\ddot{\varphi} = (c_1 - c_3)\left(\mu^2 t^2 \sin\varphi\cos\varphi + P_1\mu t\cos 2\varphi - P_1^2\sin\varphi\cos\varphi\right) + \mu(x\sin\varphi - y\cos\varphi), \tag{1.8}$$

where c_1, c_3, a_3 are corresponding elements of diagonal matrices, and $\boldsymbol{r} = (x, y, 0)$.

For a balanced body ($x = y = 0$) without an initial impulse ($P_1 = 0$) we obtain the remarkably simple equation

$$\ddot{\varphi} = kt^2\sin\varphi\cos\varphi, \quad k = \frac{\mu^2(c_1 - c_3)}{a_3}. \tag{1.9}$$

Remark. In [9,10,15] this equation is called the Chaplygin equation. In 1890, Chaplygin, being a student, obtained it together with other interesting results. However, he refrained from publishing it. The possible reason was that he could not integrate this equation explicitly. Nevertheless, this work was included in the collected works by Chaplygin, first published in his lifetime (1933, [20]).

Equation (1.9) was also obtained by Goryachev (1893) [5] and Steklov (1894) [16, 17] independently. The latter also observed the simplest properties of solutions of the equation. In particular, Steklov showed that while a body falls down, the amplitude of its oscillations with respect to the horizontal axis decreases, and the oscillation frequency grows. Steklov drew this conclusion in the supplement to his book [17]. In [17], analyzing the asymptotic behavior of a body, he made a series of inaccuracies. The Steklov problem [16, 17] about the asymptotic description of behavior of solutions of the equation was solved by Kozlov [9], who showed that under almost all initial conditions, the motion of a body approaches the uniformly accelerated fall by the wide side upward and it oscillates around the horizontal axis with the increasing frequency of order t and the decreasing amplitude of order $\frac{1}{\sqrt{t}}$. A numerical analysis of asymptotic motions with a different number of half-turns can be found in [23]. Analytic expressions for the asymptotics of a fall were obtained in various forms in [15, 23].

A phenomenon of emerging is described and studied in [7]. Under conditions of a vortex-free flow around a body it is assumed that at the initial moment the wide side of the body is horizontal and the body acquires a horizontal velocity. At subsequent moments the body begins to submerge. However, if its apparent additional mass in the transversal direction is sufficiently large, then, further, the body abruptly emerges by the narrow side upward, rising to a greater height than at the initial moment.

1.4. The motion of an axially symmetric body (circular disk). There is an important special case when system (1.6) has the additional (autonomous) linear Lagrange integral

$$M_3 = \text{const},$$

which exists under the condition of the axial symmetry of the body. We can choose the moving axes so that

$$\mathbf{A} = \operatorname{diag}(a_1, a_1, a_3), \quad \mathbf{B} = \operatorname{diag}(b_1, b_1, b_3), \quad \mathbf{C} = \operatorname{diag}(c_1, c_1, c_3), \quad \boldsymbol{r} = (0, 0, z).$$

In this case, the evolution of projections of the angular momentum on the fixed axes $\boldsymbol{N} = \big((\boldsymbol{M}, \boldsymbol{\alpha}), (\boldsymbol{M}, \boldsymbol{\beta}), (\boldsymbol{M}, \boldsymbol{\gamma})\big)$ and of the vector $\boldsymbol{n} = (\alpha_3, \beta_3, \gamma_3)$ of the axis of symmetry is described by the Hamiltonian system on $e(3)$

$$\dot{\boldsymbol{N}} = \frac{\partial \bar{H}}{\partial \boldsymbol{N}} \times \boldsymbol{N} + \frac{\partial \bar{H}}{\partial \boldsymbol{n}} \times \boldsymbol{n}, \quad \dot{\boldsymbol{n}} = \frac{\partial \bar{H}}{\partial \boldsymbol{N}} \times \boldsymbol{n}. \tag{1.10}$$

After removing of nonessential terms, the Hamiltonian (1.6) can be written in the form

$$\bar{H} = \frac{1}{2} a_1 \boldsymbol{N}^2 + b_1(P_1 N_1 - \mu t N_3) + M_3(b_3 - b_1)(P_1 n_1 - \mu t n_3) + \frac{1}{2}(c_3 - c_1)(P_1 n_1 - \mu t n_3)^2 + \mu z n_3. \tag{1.11}$$

The trajectory of the origin of the moving coordinates C can be obtained from the equations

$$\begin{aligned}
\dot{x}_1 &= b_1 N_1 + (b_3 - b_1) M_3 n_1 + P_1\big(c_1 + (c_3 - c_1) n_1^2\big) - \mu t (c_3 - c_1) n_1 n_3, \\
\dot{x}_2 &= b_1 N_2 + (b_3 - b_1) M_3 n_2 + P_1 (c_3 - c_1) n_1 n_2 - \mu t (c_3 - c_1) n_2 n_3, \\
\dot{x}_3 &= b_1 N_3 + (b_3 - b_1) M_3 n_3 + P_1 (c_3 - c_1) n_1 n_3 - \mu t \big(c_1 + (c_3 - c_1) n_3^2\big).
\end{aligned} \tag{1.12}$$

If the gravitation is balanced by the Archimedean force, then the Hamiltonian is independent of time:

$$\bar{H} = \frac{1}{2} a_1 \boldsymbol{N}^2 + b_1 P_1 N_1 + M_3(b_3 - b_1) P_1 n_1 + \frac{1}{2}(c_3 - c_1) P_1^2 n_1^2 + \mu_{\mathrm{b}} z n_3. \tag{1.13}$$

In the case of the zero initial (horizontal) impulse, i.e., if $P_1 = 0$, there is one more additional integral

$$N_3 = (\boldsymbol{M}, \boldsymbol{\gamma}) = \text{const}.$$

Moreover, for the nutation angle θ ($\gamma_3 = \cos\theta$) we obtain (by analogy with the plane-parallel motion) the nonautonomous second-order equation [20]

$$a_1^{-1}\ddot{\theta} = \frac{(M_3\cos\theta - N_3)(M_3 - N_3\cos\theta)}{\sin^3\theta} + (c_3 - c_1)\mu^2 t^2 \sin\theta\cos\theta - M_3(b_3 - b_1)\mu t\sin\theta + \mu z\sin\theta. \tag{1.14}$$

If the body begins motion from a state of rest, then $M_3 = N_3 = 0$, and we obtain

$$a_1^{-1}\ddot{\theta} = (c_3 - c_1)\mu^2 t^2 \sin\theta\cos\theta + \mu z\sin\theta.$$

1.5. An analog of the Hess case. Except for the cases considered above there is another situation when system (1.5), (1.6) admits an invariant relation similar to the Hess case in Euler–Poisson equations. For its existence it is necessary that the surface of the body be axially symmetric, and the axis of symmetry be perpendicular to the circular cross section of a gyration ellipsoid (i.e., a surface determined by the equation $(\boldsymbol{x}, \mathbf{A}\boldsymbol{x}) = 1$). Let one of axes connected with the body be directed along the axis of symmetry of the body surface, and the other two axes be directed so that $a_{23} = 0$, then in the Hamiltonian (1.6)

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & a_{13} \\ 0 & a_1 & 0 \\ a_{13} & 0 & a_3 \end{pmatrix}, \quad \mathbf{B} = \operatorname{diag}(b_1, b_1, b_3), \quad \mathbf{C} = \operatorname{diag}(c_1, c_1, c_3), \quad \boldsymbol{r} = (0, 0, z).$$

Under such a choice of the moving frame of reference, the invariant relation has the simplest form

$$M_3 = 0. \tag{1.15}$$

The equation describing the nutation angle coincides with (1.14) if $M_3 = 0$, i.e.,

$$a_1^{-1}\ddot{\theta} = \frac{N_3^2\cos^2\theta}{\sin^3\theta} + (c_3 - c_1)\mu^2 t^2 \sin\theta\cos\theta + \mu z\sin\theta. \tag{1.16}$$

The difference from the Lagrange case analyzed above appears in the equations defining the evolution of angles of precession and proper rotation. In [3], an analog of the Hess case for Eqs. (1.5), (1.6) was first pointed out.

2. THE MOTION OF AN ISOTROPIC BODY

Let us consider the simplest particular case when the equations can be solved by quadratures. The case was pointed out by Steklov [17, 18]. Here

$$\mathbf{A} = \operatorname{diag}(a_1, a_2, a_3), \quad \mathbf{B} = b\mathbf{E}, \quad \mathbf{C} = c\mathbf{E}, \quad \boldsymbol{r} = 0,$$

i.e., the added mass tensor is spherical, however, the body does not have three planes of symmetry, since $\mathbf{B} \neq 0$. (If $\mathbf{B} = 0$, then the motion is trivial: the center of gravity describes a parabola, and the motion of apexes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ is the same as in the Euler–Poinsot case.)

We isolate equations describing the evolution of angular momentum in the moving frame of reference. They are identical to those in the Euler–Poinsot case:

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \mathbf{A}\boldsymbol{M}.$$

Nevertheless, in order to determine the trajectory of the center of gravity, it is more convenient to rewrite the equations of motion in the fixed frame of reference:

$$\begin{gathered} \dot{N}_1 = b\mu t N_2, \quad \dot{N}_2 = -b\mu t N_1 - bP_1 N_3, \quad \dot{N}_3 = bP_1 N_2, \\ \dot{x}_1 = bN_1 + cP_1, \quad \dot{x}_2 = bN_2, \quad \dot{x}_3 = bN_3 - c\mu t, \end{gathered} \tag{2.1}$$

where $\boldsymbol{N} = \big((\boldsymbol{\alpha}, \boldsymbol{M}), (\boldsymbol{\beta}, \boldsymbol{M}), (\boldsymbol{\gamma}, \boldsymbol{M})\big)$ is the angular momentum in the fixed frame of reference. It is obvious that the squared angular momentum gives the integral of motion: $\boldsymbol{M}^2 = \boldsymbol{N}^2 = \text{const}$.

If the initial impulse is equal to zero: $P_1 = 0$, then the first three equations in (2.1) are integrable in terms of the elementary functions:

$$N_1 = A\sin(b\mu t^2/2 + \varphi_0), \quad N_2 = A\cos(b\mu t^2/2 + \varphi_0), \quad N_3 = \text{const},$$

where A, φ_0 are arbitrary constants. The body moves along the vertical axis with a constant acceleration: $x_3 = -\dfrac{c\mu t^2}{2}$, and the projection of the trajectory on the plane x_1, x_2 is a spiral which is described by the Fresnel integrals and converges to some fixed point on the plane.

For large times, there holds the asymptotic representation of the form

$$\begin{aligned} x_1 &= x_1^0 - \frac{A}{\mu}\frac{\cos(b\mu t^2/2 + \varphi_0)}{t} + O(t^{-3}), \\ x_2 &= x_2^0 + \frac{A}{\mu}\frac{\sin(b\mu t^2/2 + \varphi_0)}{t} + O(t^{-3}). \end{aligned}$$

If $P_1 \neq 0$, then the equations for $\boldsymbol{N}$ are nonintegrable in terms of the elementary functions. Moreover, on the plane x_1, x_2 a drift appears along the axis Ox_1 with the speed cP_1.

3. QUALITATIVE ANALYSIS OF THE PLANE-PARALLEL MOTION

It was shown above that for a special choice of the moving axes when the kinetic energy is diagonal the angle between the vertical axis and the axis connected with the body (Fig. 1) is described by Eq. (1.8), and the motion of the origin of the moving system C (Fig. 1) is described by the equations

$$\begin{aligned} \dot{X} &= (\boldsymbol{\alpha}, \mathbf{C}\boldsymbol{p}) = P_1(c_1\sin^2\varphi + c_2\cos^2\varphi) - \mu t(c_1 - c_2)\sin\varphi\cos\varphi, \\ \dot{Y} &= (\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{p}) = P_1(c_1 - c_2)\sin\varphi\cos\varphi - \mu t(c_1\cos^2\varphi + c_2\sin^2\varphi). \end{aligned} \tag{3.1}$$

Remark. Equation (1.8) corresponds to a *nonautonomous* Hamiltonian system with one degree of freedom. Such systems are studied in more detail in the case when the Hamiltonian is a periodic function of time. In the general case, they demonstrate a chaotic behavior. At the same time, as will be shown below, the dependence of angle φ on time t for system (1.8) is of asymptotic character.

First consider the "simplest" case when a balanced body ($x = y = 0$) falls without an initial impulse ($P_1 = 0$). Then, after the change $2\varphi = \theta$, Eq. (1.8) takes the form

$$\ddot{\theta} = kt^2\sin\theta, \quad k = \frac{\mu^2(c_1 - c_2)}{a_3}. \tag{3.2}$$

In what follows we assume that $c_1 > c_2$, i.e., $k > 0$, and $0 \leqslant \theta < 2\pi$.

3.1. Stationary (equilibrium) solutions. Small oscillations. Biasymptotic solutions. Equation (3.2) has the simplest "equilibrium" solutions of the form $\theta(t) = \text{const}$:

$$(1) \quad \theta = 0, \qquad (2) \quad \theta = \pi. \tag{3.3}$$

The first solution corresponds to the fall by the narrow side downward ($X = X_0$, $Y = Y_0 - \dfrac{\mu c_1 t^2}{2}$), and the second one corresponds to that by the wide side ($X = X_0$, $Y = Y_0 - \dfrac{\mu c_2 t^2}{2}$). Indeed, since $c_1^{-1} < c_2^{-1}$, the angle $\varphi = \pi n$ if the axis Ox is vertical, and $\varphi = \dfrac{\pi}{2} + \pi n$ if the axis Oy is vertical.

Linearizing Eq. (3.2) near the fixed points (3.3), we obtain

$$\begin{aligned} &(1) \quad \ddot{\xi} = kt^2\xi, && \theta = \xi, \\ &(2) \quad \ddot{\xi} = -kt^2\xi, && \theta = \pi - \xi. \end{aligned}$$

The general solution of these equations is expressed in terms of Bessel function

$$\begin{aligned} &(1) \quad \xi(t) = \sqrt{t}\left(C_1 I_{1/4}\left(\sqrt{k}t^2/2\right) + C_2 K_{1/4}\left(\sqrt{k}t^2/2\right)\right), \\ &(2) \quad \xi(t) = \sqrt{t}\left(C_1 J_{1/4}\left(\sqrt{k}t^2/2\right) + C_2 Y_{1/4}\left(\sqrt{k}t^2/2\right)\right), \end{aligned} \tag{3.4}$$

where $I_\nu(x)$, $K_\nu(x)$ are Bessel functions of the second kind, and $J_\nu(x)$, $Y_\nu(x)$ are Bessel functions of the first kind. Thus, in the linear approximation the first solution is unstable, and the second one is (asymptotically) stable with respect to ξ, but not to $\dot{\xi}$. Indeed, using the asymptotics of Bessel functions J_ν, Y_ν for large values of the argument, we find

$$\xi(t) = \frac{A\sin\left(\sqrt{k}t^2/2 + \alpha_0\right)}{\sqrt{t}} + O\big(t^{-5/2}\big), \quad A = \text{const}.$$

Consequently, the amplitude of oscillations decreases similarly to $t^{-1/2}$, and their frequency increases infinitely similarly to t. As mentioned above, this fact was noted in [9].

As shown in [9], using variational methods, one can prove that there exist two solutions $\theta(t)$, $\theta(t_0) = \theta_0$ asymptotic to the unstable position of equilibrium ($\theta = 0$), which approach it from different sides. In addition, because of the invariance of Eq. (3.2) with respect to the change $t \to -t$, there exists a solution $\theta_*(t)$ with the initial data $\theta_*(0) = \pi$ for which [9]

$$\theta_*(t) + \theta_*(-t) = 2\pi, \quad \lim_{t\to-\infty} \theta_*(t) = 0, \quad \lim_{t\to+\infty} \theta_*(t) = 2\pi.$$

Thus, the solution $\theta_*(t)$ is biasymptotic (there is also a similar biasymptotic solution passing around the circle θ $[0, 2\pi]$ contrariwise). Here the body makes one half-turn. Its trajectory is described by Eqs. (3.1) and shown in Fig. 2a. Note that the upper point of the trajectory is a cusp point: the equation of the curve near this point has the form $Y = \lambda X^{2/3}$, $\lambda = \text{const}$. In Fig. 2b the change of angle φ for this biasymptotic solution is shown.

The existence of biasymptotic trajectories with an arbitrary number of half-turns was proved in [22].

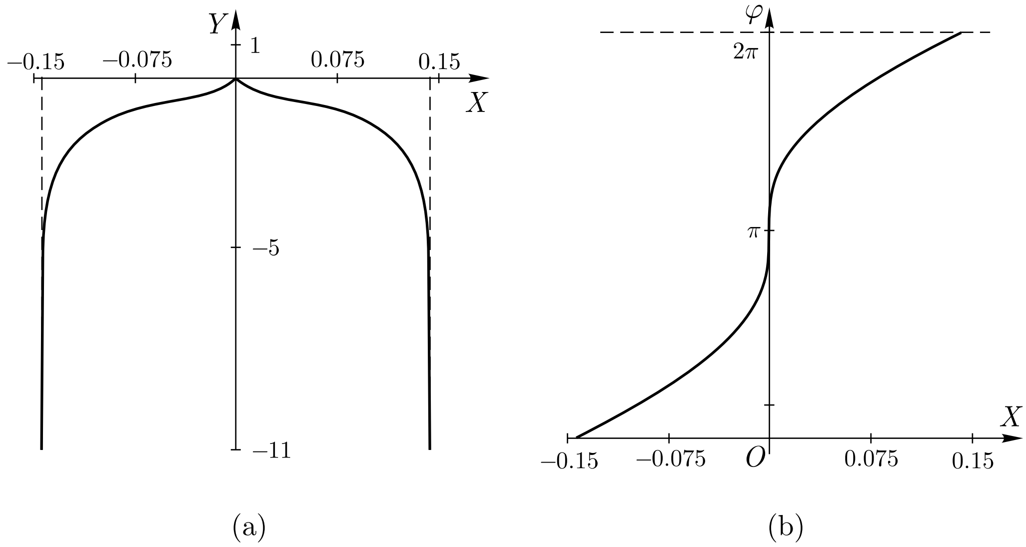


Fig. 2. The trajectory of the body and the value of angle φ depending on the coordinate X for the biasymptotic solution with $k = 1$, $\frac{a_3}{\mu} = 0.1$ $(P_1 = 0)$; the upper point of the trajectory is singular (see text).

3.2. The asymptotic behavior of solutions of the Chaplygin equation. As shown in [9] (the idea of the proof in a more general case is presented below), for all solutions of the equation, either $\theta \to 0$ or $\theta \to \pi$ as $t \to \pm\infty$ (i.e., the asymptotic motion of the body is the fall by the wide or narrow side forward).

There is a hypothesis stated by Kozlov [9] that the measure of trajectories which tend to unstable equilibrium state $\theta = 0 (\mathrm{mod} 2\pi)$ as $t \to \pm\infty$, is equal to zero, and thus almost all trajectories tend to $\lim\limits_{t\to\infty} \theta(t) = \pi (\mathrm{mod} 2\pi)$ (i.e., to the fall by the wide side forward).

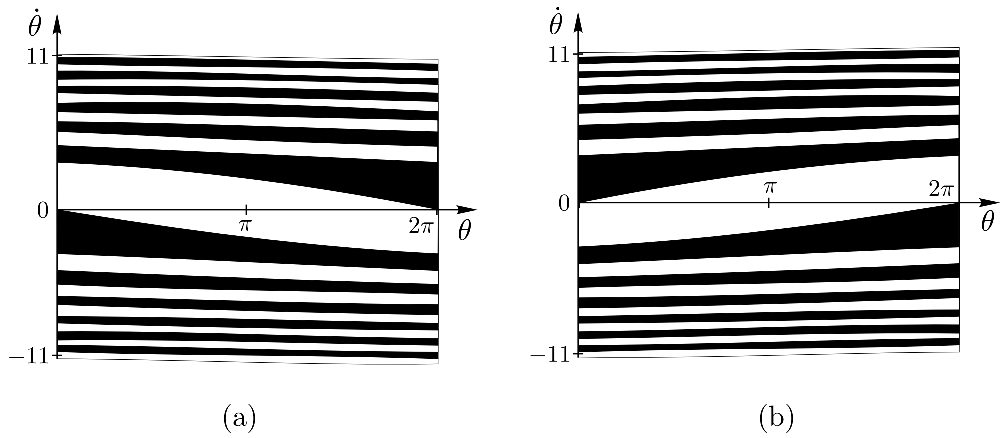


Fig. 3. Domains of the phase plane corresponding to the initial conditions with $t_0 = 0$, when the body makes the same number of half-turns while t changes from 0 to $+\infty$ in the case (a), and while t changes from $-\infty$ to 0 in the case (b) ($k = 1$, the white colour corresponds to an even number of half-turns, the black one corresponds to an odd number).

3.3. Numerical analysis. On the basis of the statement about the asymptotic behavior, a numerical analysis can be applied to Eq. (3.2) [23]. For this, on the phase plane $(\theta, \dot\theta)$ (more precisely, on the cylinder $\theta \times \dot{\theta} \in [0, 2\pi) \times (-\infty, +\infty)$) at the initial moment of time $t = t_0$ domains are constructed where the body makes the same number of half-turns as $t \to +\infty$ (or as $t \to -\infty$) before it will be "attracted" to the solution $\theta = \pi$. As seen in Fig. 3a, these domains are located regularly, moreover, their width decreases while $|\dot{\theta}|$ increases so that for large initial $|\dot{\theta}|$ only the probability of the fall of the body by the "upper" or "under" side as $t \to +\infty$ makes sense. Boundaries of the domains are filled by initial conditions corresponding to motions asymptotically approaching the unstable positions of equilibrium $\theta = 0,\ 2\pi$. Analogously, one can construct domains corresponding to the same number of half-turns as $t \to -\infty$ (Fig. 3b), moreover, the domains for $t \to +\infty$ turn out to be the mirror images about the line $\theta = \pi$ of the domains for $t \to -\infty$. If we overlay these domains, their boundaries are intersected at points from the line $\theta = \pi$. Biasymptotic solutions of Eq. (3.2) with different number of half-turns correspond to them.

Remark. On the cylinder $\theta \times \dot{\theta} \in [0, 2\pi) \times (-\infty, +\infty)$, all boundaries of the domains are glued together in one smooth curve similar to a screw line whose step decreases when $|\dot{\theta}|$ increases. Domains with an even number of half-turns lie on one side of this line, domains with an odd number lie on the other.

Thus, the numerical computations confirm the conjecture that for almost all solutions $\theta(t) \xrightarrow[t \to \pm\infty]{} \pi$, moreover, in the three-dimensional space t, θ, $\dot{\theta}$, solutions asymptotically approaching unstable equilibrium $\theta = \pi$ fill two-dimensional surfaces. In addition, there is also a countable set of biasymptotic solutions that differ by the number of half-turns made when t varies from $-\infty$ to $+\infty$.

In Fig. 4, the trajectories of a body for the biasymptotic motions with one and three half-turns is shown.

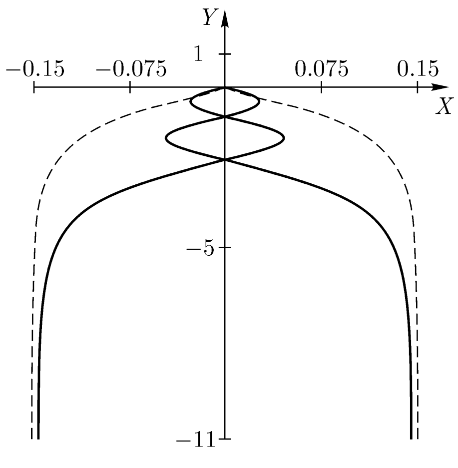


Fig. 4. The trajectory of a body in the case of biasymptotic solutions with one (the dotted line) and three (the solid line) half-turns for $k > 0$, $a_3/\mu = 0.1$. There is a singularity in the upper point of the trajectory.

3.4. The trajectory of the body. Substituting the asymptotic decomposition for small oscillations (3.4) into Eq. (3.1), after integration we obtain the asymptotic representation for the trajectory of the motion in the form

$$X(t) = A\frac{\cos\bigl(\sqrt{k}t^2/2 + \theta_0\bigr)}{\sqrt{t}} + O\bigl(t^{-3/2}\bigr), \quad Y(t) = -\mu c_2 t^2 + O\bigl(t^{-1/2}\bigr),$$

where A, θ_0 are some constants. Therefore, the trajectory of the motion for large times is close to the sinusoid with the constant step $\Delta y = \dfrac{\pi\mu c_2}{\sqrt{k}}$ and a decreasing amplitude [9]. (Step ΔY is calculated between two consecutive zeros of the function $X(t)$.) The typical trajectory is shown in Fig. 5.

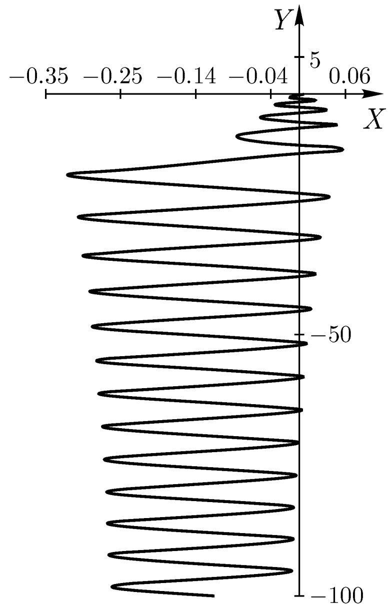


Fig. 5. The typical form of trajectory of a body falling without an initial impulse.

3.5. General case ($P_1 \neq 0$)**.** Now let us review the main qualitative features of the behavior of system (1.8), (3.1) in the general case. If $P_1 \neq 0$, there are no longer time-independent solutions similar to (3.3). A statement about the asymptotic behavior also holds in this case. According to this statement, for any solution $\varphi(t)$ of Eq. (1.8) we have

$$1. \lim_{t\to+\infty} \varphi(t) = \pi n \quad \text{or} \quad 2. \lim_{t\to\infty} \varphi(t) = \frac{\pi}{2} + \pi n, \quad n \in \mathbb{Z}.$$

It also seems that *almost all solutions of the equation tend to a solution of the form* $\varphi(t) = \dfrac{\pi}{2} + \pi n$ (i.e., the motion of the body approaches the fall by the wide side downward) [9]. Numerical experiments confirm this.

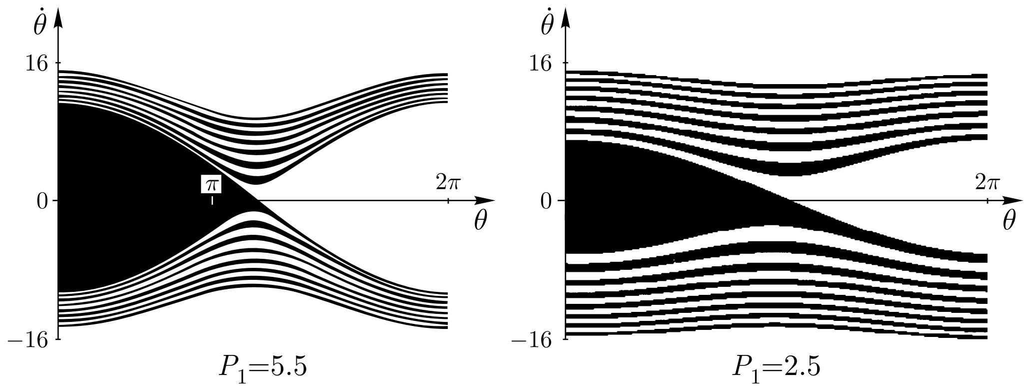


Fig. 6.

Similarly, we can perform a computer analysis, considering those domains on the phase plane at the initial moment $t = t_0$ to which there corresponds the same number of half-turns before the trajectory will be attracted to the solution $\varphi = \frac{\pi}{2}$ as $t \to +\infty$ (Fig. 6). The boundaries of these domains are filled with asymptotic solutions. As in the case $P_1 = 0$, domains corresponding to a different number of half-turns for $t_0 = 0$ and $t \to -\infty$ turn out to be the mirror image of the domains for $t_0 = 0$ and $t \to +\infty$ about the line $\varphi = \frac{\pi}{2}$. The intersection points of boundaries of the domains as $t \to +\infty$ and $t \to -\infty$ correspond to biasymptotic solutions.

The typical form of the trajectory of a body thrown at an angle to the horizon is shown in Fig. 7. In Fig. 8 the trajectories are given in the case of biasymptotic motions with one and three half-turns of the body.

As shown in [9], in the general case, the asymptotic trajectory of the body is a parabola:

$$X(t) = -P_1 t + o(t), \quad Y(t) = -\frac{\mu t^2}{2c_3} + o(t^2).$$

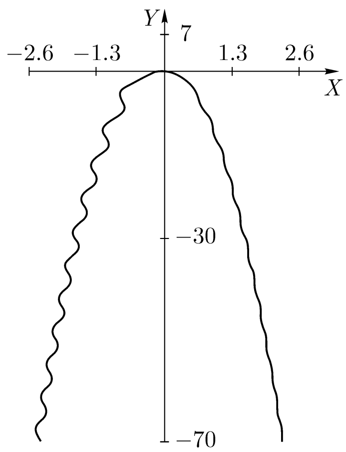


Fig. 7. The typical form of the trajectory of a rigid body thrown at an angle to the horizon.

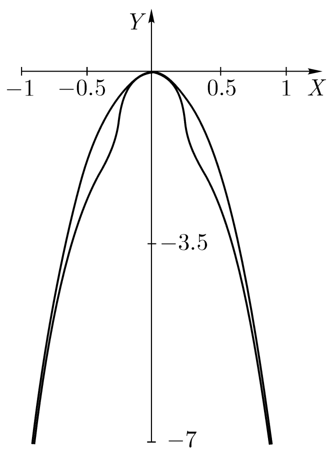


Fig. 8. The trajectories of a body for biasymptotic motions with one (the upper curve) and three half-turns.

4. A BODY WITH THREE PLANES OF SYMMETRY

As above (for the plane-parallel motion), before studying the general system (1.6), we consider in detail the special case of motion without an initial impulse ($P_1 = 0$) under the additional constraints

$$\mathbf{B} = 0, \quad \boldsymbol{r} = 0. \tag{4.1}$$

Here we obtain a nonautonomous Hamiltonian system (on $e(3)$) for $\boldsymbol{M}$, $\boldsymbol{\gamma}$ with the Hamiltonian

$$\bar{H} = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + \frac{1}{2}\mu^2 t^2 (\mathbf{C}\boldsymbol{\gamma}, \boldsymbol{\gamma}). \tag{4.2}$$

(In the general case, we can assume that $\mathbf{A}$ is diagonal and $\mathbf{C}$ is arbitrary symmetric.)

4.1. Time-independent (equilibrium) solutions and "normal oscillations." The equations of motion for system (4.2) have the form

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \mathbf{A}\boldsymbol{M} + \mu^2 t^2 \boldsymbol{\gamma} \times \mathbf{C}\boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \mathbf{A}\boldsymbol{M} \tag{4.3}$$

and admit the simplest solutions of the form

$$\boldsymbol{M} = 0, \quad \boldsymbol{\gamma} = \pm \boldsymbol{\xi}_i, \quad i = 1, 2, 3, \tag{4.4}$$

where $\boldsymbol{\xi}_i$ are eigenvectors of the matrix $\mathbf{C}$ (for $\mathbf{C}$ degenerate, there are infinitely many eigenvectors $\boldsymbol{\xi}_i$).

Linearizing system (4.3) near solution (4.4), by linear transformations of coordinates one can reduce the equations of motion to the form of "normal oscillations"

$$\ddot{x}_k + t^2 \omega_k x_k = 0, \quad k = 1, 2, \tag{4.5}$$

where x_k are corresponding local coordinates near the fixed points $\boldsymbol{\gamma} = \boldsymbol{\xi}_i$. Solutions of system (4.5) can be expressed in Bessel functions (see (3.4)). If all eigenvalues of $\mathbf{C}$ are different, then to the local minimum of the function $V(\boldsymbol{\gamma}) = \dfrac{1}{2}(\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{\gamma})$ there corresponds an (asymptotically) stable solution of system (4.5) whose asymptotic form for large t is (3.4). The local minimum is defined by some eigenvector of the system. Unstable (now in the linear approximation) solutions correspond to other two eigenvectors.

4.2. The asymptotic behavior of solutions. It turns out that similarly to the plane-parallel case, under arbitrary initial conditions the vector $\boldsymbol{\gamma}$ tends to one of eigenvectors of the matrix $\mathbf{C}$. Indeed, in [11] it is shown that *for any solution* $\boldsymbol{\gamma}(t)$ *of Eqs.* (4.3)

$$\lim_{t\to\infty} V\bigl(\boldsymbol{\gamma}(t)\bigr) = \mathcal{E}_c,$$

where $V(\boldsymbol{\gamma}) = \dfrac{1}{2}(\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{\gamma})$*, and* $\mathcal{E}_c$ *is a critical value of function* $V(\boldsymbol{\gamma})$*.*

The proof of this statement is based on a representation of Eqs. (4.3) in terms of the new time and new variables. Let us change time and variables by the formulas

$$\frac{1}{2}t^2 = \tau, \quad t\boldsymbol{M} = \boldsymbol{m};$$

here the equations of motion have the form

$$\frac{d\boldsymbol{m}}{d\tau} = -\frac{1}{2\tau}\boldsymbol{m} + \boldsymbol{m} \times \mathbf{A}\boldsymbol{m} + \mu^2 \boldsymbol{\gamma} \times \mathbf{C}\boldsymbol{\gamma}, \quad \frac{d\boldsymbol{\gamma}}{d\tau} = \boldsymbol{\gamma} \times \mathbf{A}\boldsymbol{m}. \tag{4.6}$$

It is easy to show that $\operatorname{div} \boldsymbol{v} = -\dfrac{1}{2\tau}$, i.e., in fact, system (4.6) describes the Kirchhoff equations with dissipation decreasing with time. Consider the energy of "unperturbed" system

$$\mathcal{E} = \frac{1}{2}(\boldsymbol{m}, \mathbf{A}\boldsymbol{m}) + \frac{1}{2}\mu^2(\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{\gamma}). \tag{4.7}$$

Calculating the derivative $\mathcal{E}$ along solutions (4.6), we find

$$\frac{d\mathcal{E}}{d\tau} = -\frac{(\boldsymbol{m}, \mathbf{A}\boldsymbol{m})}{2\tau}.$$

From this equality it easily follows that

(1) $\mathcal{E} \xrightarrow[t\to\infty]{} \mathcal{E}_* = \text{const}$;

(2) the integral $I = \displaystyle\int_{\tau_0}^{\infty} \frac{\bigl(\boldsymbol{m}(\tau), \mathbf{A}\boldsymbol{m}(\tau)\bigr)}{2\tau} d\tau$ converges.

The proof is reduced to showing that $\mathcal{E}_* = \mathcal{E}_c$ is the critical value of (4.7), and hence, of $V(\boldsymbol{\gamma})$. It turns out that the assumption $\mathcal{E}_* \neq \mathcal{E}_c$ contradicts the convergence of the integral I.

For the fall of an arbitrary body with three planes of symmetry, there also exists the conjecture [11] that $\mathcal{E}_* = \mathcal{E}_c^{\min}$ *for almost all solutions* $\boldsymbol{\gamma}$ *of Eqs.* (4.3). Thus, as $t \to \infty$, the body almost always tends to occupy a position in space such that the axis corresponding to the maximal added mass becomes vertical.

4.3. Computer analysis. The statement about the asymptotic behavior formulated above leads to the natural question about the structure of domains (basins of attraction) corresponding to different asymptotic modes as $t \to \pm\infty$ in the space of initial conditions. Let us choose $t_0 = 0$ and parametrize the joint four-dimensional level of the integrals

$$(\boldsymbol{M}, \boldsymbol{\gamma}) = c = \text{const}, \quad \boldsymbol{\gamma}^2 = 1 \tag{4.8}$$

by Andoyer variables (L, G, l, g) and fix the surface of initial conditions for $t_0 = 0$ by the equations

$$g = g_0, \quad E = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) = \text{const}. \tag{4.9}$$

Depending on the side by which the body falls as $t \to \infty$, we shall paint the point on this surface with a corresponding color. The typical picture is given in Figs. 9, 10.

We see that the body falls so that the axis corresponding to the maximal apparent additional mass is vertical, i.e., it falls either by one wide side downward, or by the other one. This confirms the conjecture formulated above. In addition, generally, the boundary of these domains is fractal, i.e., the surface pattern repeats in smaller parts.

Thus, by analogy with the integrable and nonintegrable (regular and chaotic) systems, the plane-parallel case can be called integrable, and the general case of system (4.2), (4.3) can be called nonintegrable. Indeed, in the plane-parallel case the boundaries of domains corresponding to the different orientations of the body are regular, but in system (4.2), (4.3) they are fractal. We shall show below that if system (4.3) has one more additional integral (the Lagrange integral), the boundaries of domains also become regular.

The fractal structure of boundaries separating the different types of behavior as $t \to \infty$ is closely connected with probabilistic effects arising when descripting asymptotic motions. Indeed, for complex distribution of initial conditions corresponding to different types of asymptotic behavior, under specific (given) initial conditions, the asymptotic behavior becomes unpredictable and only probabilistic description makes sense. This is a kind of probabilistic chaos generated by the structure of initial conditions. The probabilistic description was also proposed by A. I. Neishtadt when he studied the motion around a fixed point of a rigid body under the action of constant and linear (with respect to the angular velocity) dissipative moments [14]. It turned out that for small values of these moments, the dynamics of the system has probabilistic nature. In [14] the explicit formulas for probabilities realizing the evolution of the system to a uniform rotation are obtained. The straight generalization of analytic results [14] to system (4.3), (4.6) is connected with substantial difficulties by virtue of the larger dimension of this system and the dependence of the "dissipation parameter" ε on time: $\varepsilon \sim \dfrac{1}{\tau}$.

Remark. The behavior of a heavy body in a fluid substantially differs from its inertial motion, which is described by Kirchhoff equations. In general case, the last system is nonintegrable [1, 12] and shows the typical chaotic behavior (the Hamiltonian chaos) [2, 21].

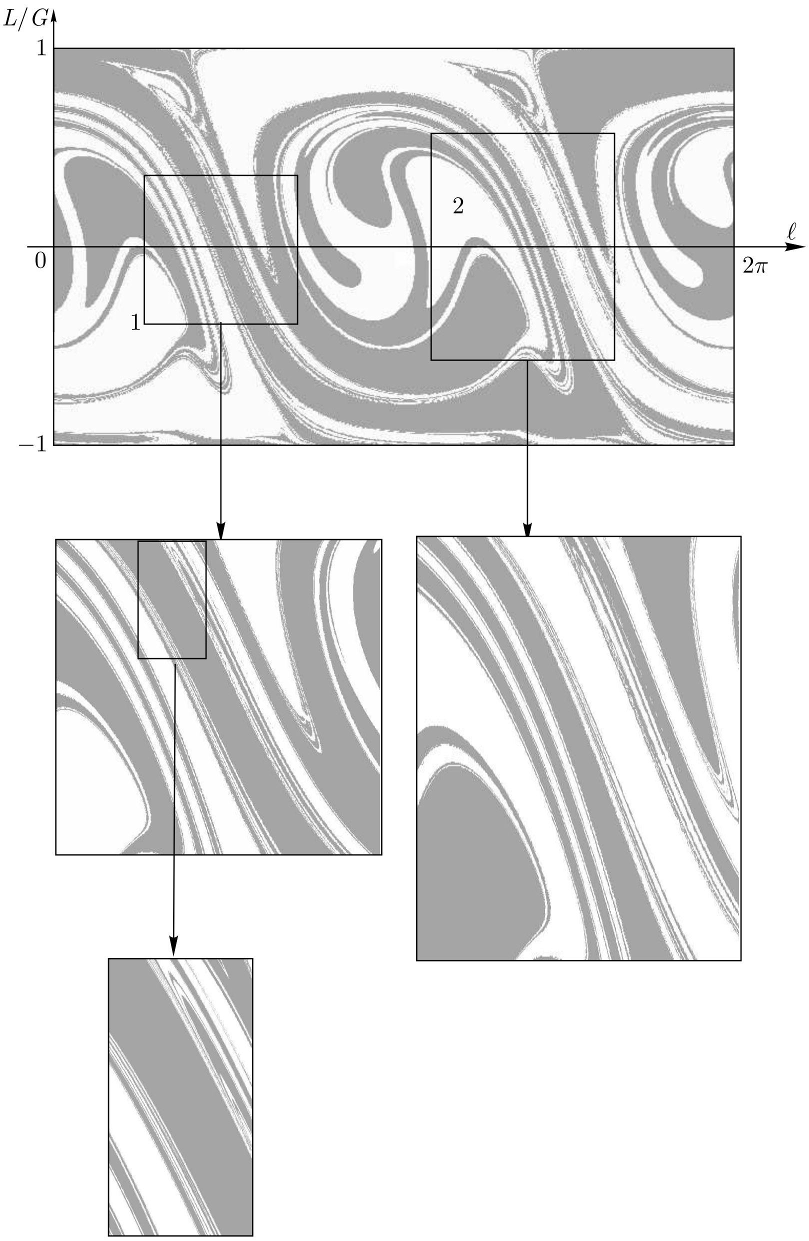
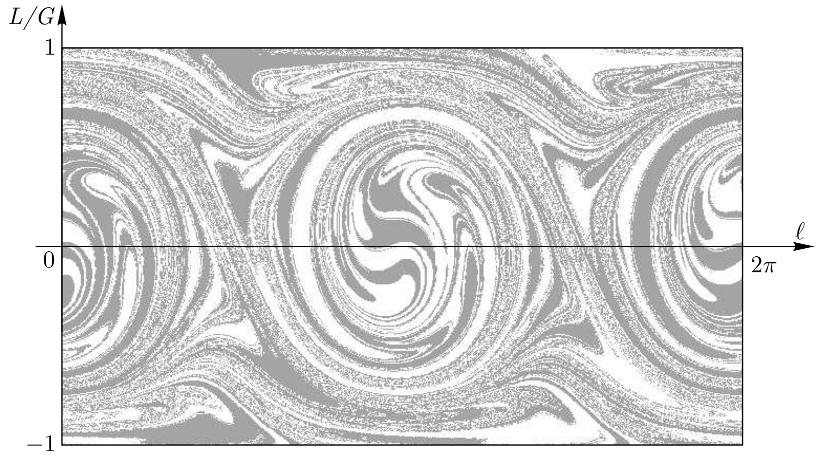
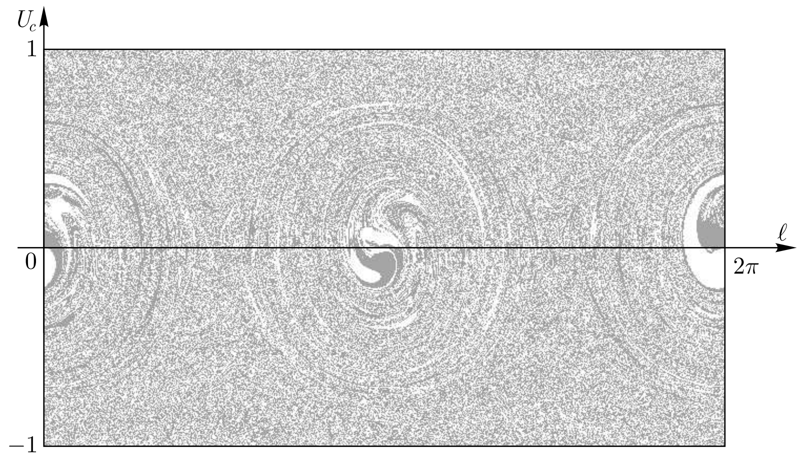


Fig. 9. The typical picture of domains corresponding to two different limit orientations of the body (where the eigenvector corresponding to the greatest added mass is vertical, two colors correspond to its two possible directions) as $t \to +\infty$. At a four-dimensional level of the first integrals, the given two-dimensional surface is defined by Eqs. (4.8), (4.9). The values of the system parameters are $\mathbf{A} = \mathrm{diag}(1.8, 1.5, 2)$, $\mathbf{C} = \mathrm{diag}(0.5, 2.9, 1.4)$, $\mu = 1$, $(\boldsymbol{M}, \boldsymbol{\gamma}) = 1$, $E_0 = 7$.



(a) $E_0 = 20, \quad t_0 = 0.6$



(b) $E_0 = 70, \quad t_0 = 0.3$

Fig. 10. The typical pattern on the surface of initial conditions according to the behavior of the system as $t \to \infty$ when the initial energy and the initial moment t_0 increase. The values of the system parameters are $\mathbf{A} = \mathrm{diag}(1.8, 1.5, 2)$, $\mathbf{C} = \mathrm{diag}(0.5, 2.9, 1.4)$, $\mu = 1$, $(\boldsymbol{M}, \boldsymbol{\gamma}) = 1$.

5. THE FALL OF A BODY WITH SCREW SYMMETRY: STEKLOV SOLUTIONS AND THEIR STABILITY

For the general case $P_1 \neq 0$, $\mathbf{B} \neq 0$ of system (1.6), after the changes

$$\frac{1}{2}t^2 = \tau, \quad \boldsymbol{M} = t\boldsymbol{m}$$

we obtain the equations of motion in the form

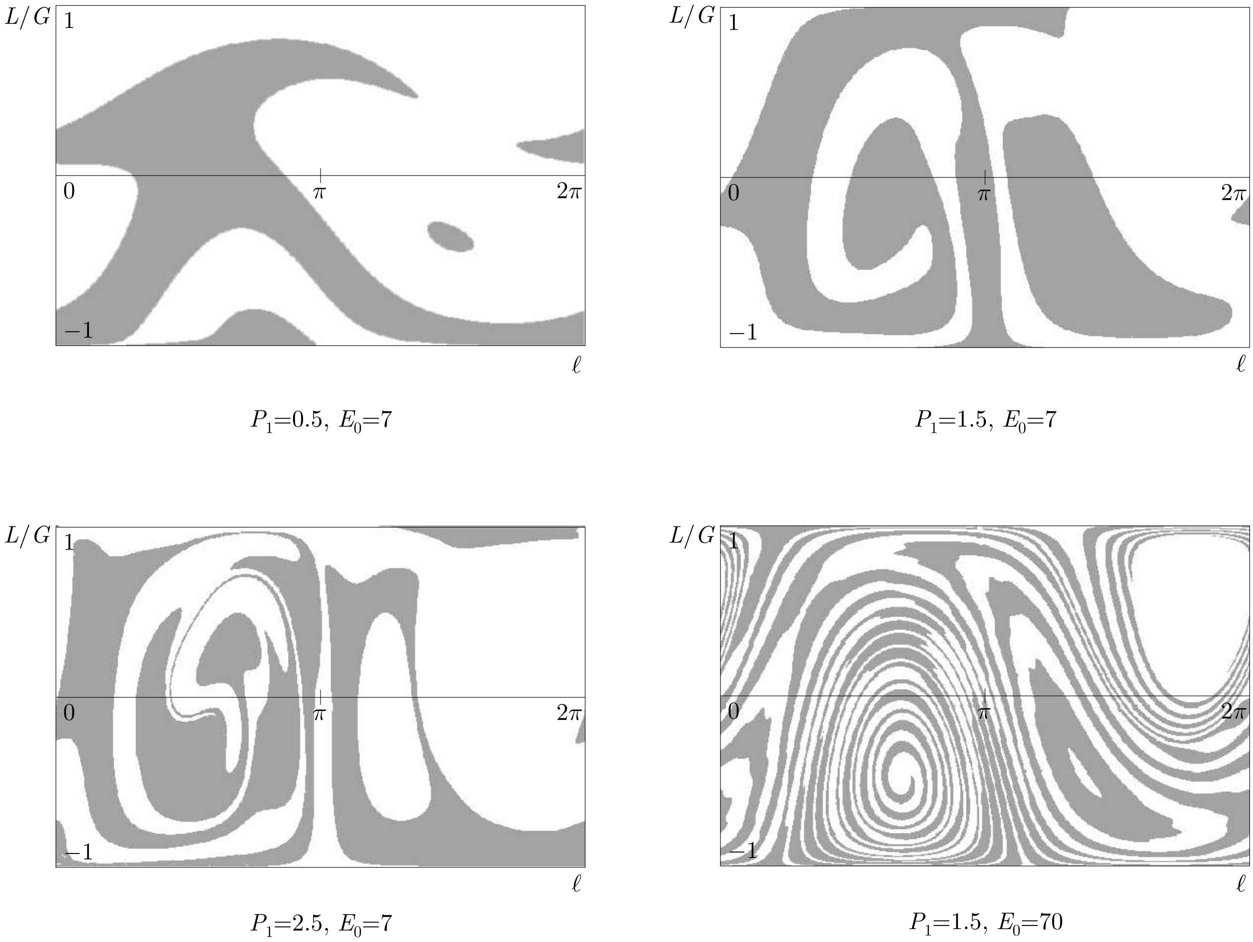


Fig. 11. An analog of the Lagrange case, i.e., the case of existence of the integral $M_3 = \mathrm{const}$. The structure of the basin of attraction is regular ($b_1 = 0.3$, $b_3 = 1.7$, $c_1 = 2.9$, $c_3 = 1.4$, $\mu = 1$).

$$\begin{gathered}\frac{d\boldsymbol{m}}{d\tau} = -\frac{1}{2\tau}\boldsymbol{m} + \boldsymbol{m}\times\frac{\partial H}{\partial \boldsymbol{m}} + \boldsymbol{\alpha}\times\frac{\partial H}{\partial \boldsymbol{\alpha}} + \boldsymbol{\gamma}\times\frac{\partial H}{\partial \boldsymbol{\gamma}},\\ \frac{d\boldsymbol{\alpha}}{d\tau} = \boldsymbol{\alpha}\times\frac{\partial H}{\partial \boldsymbol{m}}, \quad \frac{d\boldsymbol{\gamma}}{d\tau} = \boldsymbol{\gamma}\times\frac{\partial H}{\partial \boldsymbol{m}},\\ H = H_0 + \frac{1}{\sqrt{2\tau}}H_1 + \frac{1}{2\tau}H_2,\\ H_0 = \frac{1}{2}(\boldsymbol{m}, \mathbf{A}\boldsymbol{m}) - \mu(\mathbf{B}\boldsymbol{m}, \boldsymbol{\gamma}) + \frac{\mu^2}{2}(\mathbf{C}\boldsymbol{\gamma}, \boldsymbol{\gamma}),\\ H_1 = P_1(\mathbf{B}\boldsymbol{m}, \boldsymbol{\alpha}) - P_1\mu(\mathbf{C}\boldsymbol{\alpha}, \boldsymbol{\alpha}), \quad H_2 = \frac{P_1}{2}(\boldsymbol{\alpha}, \mathbf{C}\boldsymbol{\alpha}) + \mu(\boldsymbol{r}, \boldsymbol{\gamma}).\end{gathered} \tag{5.1}$$

Now, the differentiation of energy along the system gives

$$\begin{gathered}\frac{dH}{d\tau} = -\frac{(\boldsymbol{m}, \mathbf{A}\boldsymbol{m})}{2\tau} + \frac{1}{2\tau}\left(\mathbf{B}\boldsymbol{m}, \mu\boldsymbol{\gamma} - \frac{2P_1}{\sqrt{2\tau}}\boldsymbol{\alpha}\right) + \frac{W_1}{(2\tau)^{3/2}} + \frac{W_2}{(2\tau)^2},\\ W_1 = -P_1\mu(\mathbf{C}\boldsymbol{\alpha}, \boldsymbol{\gamma}), \quad W_2 = P_1^2(\mathbf{C}\boldsymbol{\alpha}, \boldsymbol{\alpha}) + 2\mu(\boldsymbol{r}, \boldsymbol{\gamma}).\end{gathered} \tag{5.2}$$

For this system, the asymptotic principles of motion formulated in the previous sections are no longer valid. Moreover, complex attractive regimes of motion different from translational motions

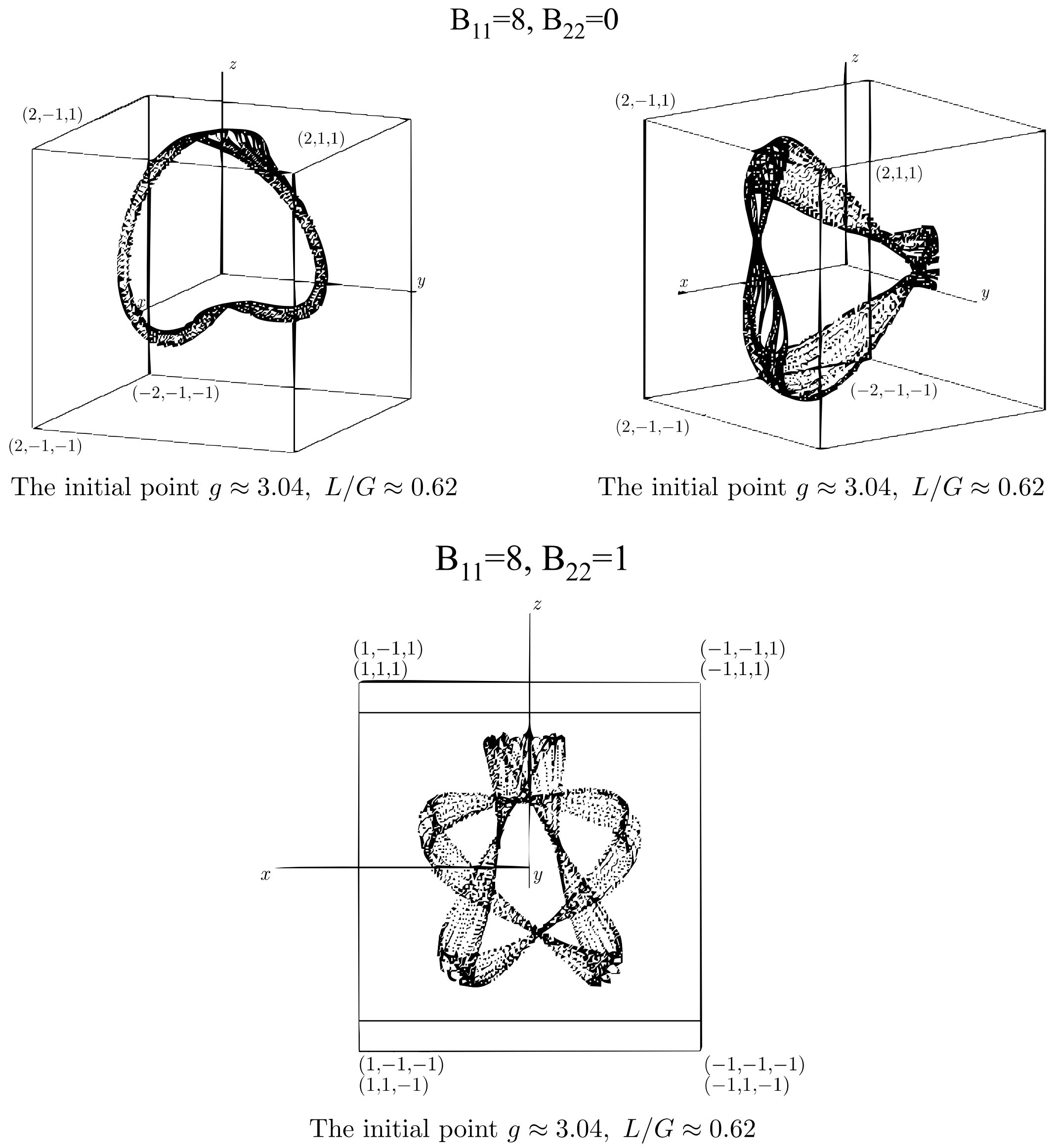


Fig. 12. The typical form of limiting sets in the case of fall of a body with screw symmetry ($A_{11} = 1$, $A_{22} = 1.2$, $A_{33} = 2$, $C_{11} = 1.6$, $C_{22} = 0.1$, $C_{33} = 0$, $P_1 = 0$, $\mu = 1$, $x = y = z = 0$, $E_0 = 3$, $g_0 = \frac{\pi}{2}$).

exist as $t \to \infty$. First of all, we consider stability conditions (with $\mathbf{B} \neq 0$) for the partial solutions of Eqs. (5.1) corresponding to uniformly accelerated rotations and find domains of values of parameters for which all these solutions lose their stability (and more complex modes become stable). Further, we also consider the case of the zero initial impulse $P_1 = 0$.

5.1. The linear stability of Steklov solutions. For $P_1 = 0$ we separate the equations for $\boldsymbol{m}$, $\boldsymbol{\gamma}$, and the area integral can be represented in the form

$$(\boldsymbol{m}, \boldsymbol{\gamma}) = \frac{\sigma}{\sqrt{2\tau}}, \quad \sigma = \text{const}, \tag{5.3}$$

i.e., $(\boldsymbol{M}, \boldsymbol{\gamma}) = \sigma$.

If, in addition, $\boldsymbol{r} = 0$ and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are simultaneously diagonalizable, then Eqs. (5.1) have partial solutions similar to the time-independent solutions of (4.4). In the basis of eigenvectors of matrices **A**, **B**, **C** we have

$$\gamma_k = \pm 1, \quad \gamma_i = \gamma_j = 0, \quad m_k = \pm\frac{\sigma}{\sqrt{2\tau}}, \quad m_i = m_j = 0, \quad i \neq j \neq k \neq i; \tag{5.4}$$

then there exist six partial solutions in total. Here the body falls so that its axis Oe_k remains vertical, and the angular velocity of rotation around it is determined by the relation

$$\Omega^{(k)} = -\mu b_k t + \sigma a_k,$$

i.e., the rotation of the body is uniformly accelerated. The velocity of the origin of the moving coordinate system in the moving axes is defined by the expression $\boldsymbol{v} = \dfrac{\partial H}{\partial \boldsymbol{p}} = (\sigma\mathbf{B} - \mu t\mathbf{C})\boldsymbol{\gamma}$, whence, using (1.2), we find

$$x_i = \text{const}, \quad x_j = \text{const}, \quad x_k = -\mu c_k \frac{t^2}{2} + \sigma b_k t + \text{const},$$

i.e., the motion of the origin along the vertical axis is uniformly accelerated similarly to the free fall of the body. These uniformly accelerated motions were found out by Steklov [18] (1895) and Chaplygin [20] (1900). In what follows we call them the Steklov solutions.

Similarly to solutions (4.4), solutions (5.4) are always unstable in the whole phase space (with respect to variables $\boldsymbol{M}$, $\boldsymbol{\gamma}$). This instability was pointed out by Steklov [18]. At the same time, the stability with respect to positional variables $\boldsymbol{\gamma}$ depends on parameters of the system and requires the special consideration.

To investigate the stability of solutions of (5.4) we choose the new variables

$$v_i = \frac{d\gamma_i}{d\tau}, \quad v_j = \frac{d\gamma_j}{d\tau}, \quad i \neq j \neq k \neq i, \tag{5.5}$$

adding the area integral (5.3) to these equations, we express variables m_i, m_j, m_k in terms of v_i, v_j, σ. Using the relation $\gamma_k = \pm 1 \mp \frac{1}{2}(\gamma_i^2 + \gamma_j^2)$ near solutions of (5.4), we obtain linearized equations for new variables in the form

$$\begin{aligned} &\frac{d\gamma_i}{d\tau} = v_i, \quad \frac{d\gamma_j}{d\tau} = v_j, \\ \frac{dv_i}{d\tau} &= -a_i^{-1} a_j \varkappa_i^{(k)} \gamma_i + a_i^{-1} a_j \mu\big(b_i - b_k + a_i(b_j - b_k)\big) v_j \\ &\quad + \frac{\sigma}{\sqrt{2\tau}} a_i^{-1}\big(\mu a_j a_k (b_i - b_k)\gamma_i + (a_i a_k + a_j a_k - a_i a_j) v_j\big) \\ &\quad - \frac{1}{2\tau}\big(v_i - \sigma^2 a_i^{-1} a_j a_k (a_k - a_i)\gamma_j + \mu(b_j - b_k)\gamma_j\big), \\ \frac{dv_j}{d\tau} &= \dots, \\ &\varkappa_i^{(k)} = \mu^2\big(a_i(c_i - c_k) - (b_i - b_k)^2\big), \end{aligned} \tag{5.6}$$

where the expression for $\dfrac{dv_j}{d\tau}$ is obtained by the change of indices $i \leftrightarrow j$.

We use theorems from [4] about the behavior of the solutions for linear systems of the form $\dfrac{d\boldsymbol{x}}{d\tau} = (\mathbf{A} + \mathbf{V}(\tau))\,\boldsymbol{x}$, where $\int\limits_{\tau_0}^{\infty} |V'(\tau)|d\tau < \infty$ and $V(\tau) \to 0$ as $t \to \infty$. Applying them, we conclude that eigenvalues of linear system (5.6) are expanded in power series of variable $\tau^{-1/2}$

$$\lambda_k(\tau) = \lambda_k^{(0)} + \frac{\lambda_k^{(1)}}{\sqrt{\tau}} + \frac{\lambda_k^{(2)}}{\tau} + \dots,$$

and the inequalities $\mathrm{Re}\lambda_k^{(0)} \leqslant 0$, where $\lambda_k^{(0)}$ are eigenvalues of system (5.6) for $\tau = \infty$, are the necessary conditions for stability of the system (similarly, $\mathrm{Re}\lambda_k^{(0)} > 0$ are the sufficient conditions for instability). To determine them, we can obtain the (biquadratic) characteristic polynomial

$$\lambda^4 - \lambda^2(\varkappa_i^{(k)} + \varkappa_j^{(k)} - \varkappa_k^{(k)}) + \varkappa_i^{(k)}\varkappa_k^{(k)} = 0. \tag{5.7}$$

Thus, the necessary condition for stability of solutions of (5.4) is that polynomial (5.7) has pure imaginary roots (more precisely, it is the condition of absence of exponential instability with respect to τ). Hence, we find the corresponding constraints on the parameters

$$\begin{gathered} \varkappa_i^{(k)} \cdot \varkappa_j^{(k)} > 0, \quad \varkappa_i^{(k)} + \varkappa_j^{(k)} - \varkappa_k^{(k)} < 0, \\ D = (\varkappa_i^{(k)})^2 + (\varkappa_j^{(k)})^2 + (\varkappa_k^{(k)})^2 - 2\varkappa_i^{(k)}\varkappa_j^{(k)} - 2\varkappa_i^{(k)}\varkappa_k^{(k)} - 2\varkappa_j^{(k)}\varkappa_k^{(k)} > 0. \end{gathered} \tag{5.8}$$

Let us now study in detail the stability of each solution depending on the parameters. Without loss of generality, we put $k = 3$ and $\mu = 1$, $c_3 = 0$, $b_3 = 0$ (the use of integrals $\boldsymbol{\gamma}^2 = 1$ and $(\boldsymbol{M}, \boldsymbol{\gamma}) = \mathrm{const}$ provides the fulfilment of the last two conditions). We fix a_1, a_2, a_3, c_1, c_2 and construct on the plane of parameters b_1, b_2 the domains where inequalities (5.8) hold. In this case, relations (5.8) take the form

$$\begin{gathered} (a_1c_1 - b_1^2)(a_2c_2 - b_2^2) > 0, \quad \Phi = a_1c_2 + a_2c_1 + 2b_1b_2 > 0, \\ D = (a_1c_2 - a_2c_1)^2 + 4(a_1b_2 + a_2b_1)(c_2b_1 + c_1b_2) > 0. \end{gathered} \tag{5.9}$$

It is easy to show that there are three qualitatively different cases:

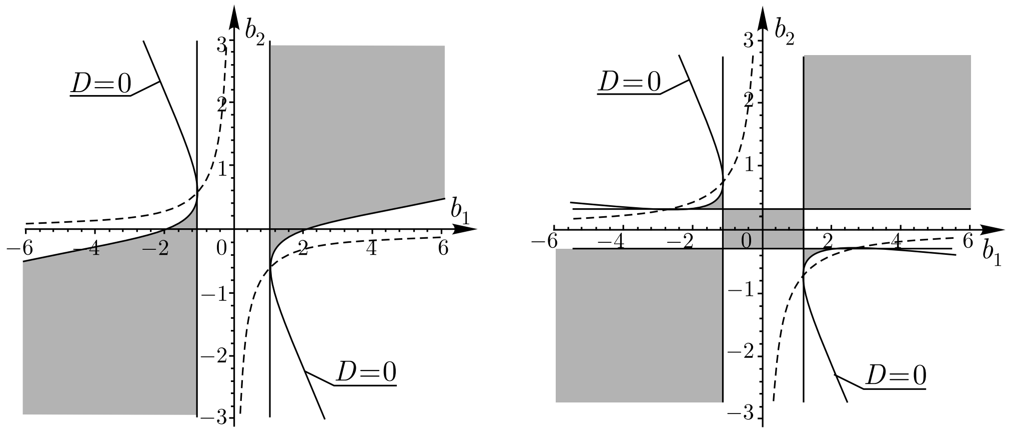
1. $c_3 = 0 > c_1 > c_2$ (i.e., $c_1 < 0$ and $c_2 < 0$); in this case, on the plane b_1, b_2 there are no domains where inequalities (5.8) hold. We can show that there will be either two pairs of real solutions of Eq. (5.7) or four complex solutions.

2. $c_1 > c_3 = 0 > c_2$ (i.e., $c_1 > 0$ and $c_2 < 0$); in this case, the domains given by relations (5.9) are located between the lines $b_1 = \pm\sqrt{a_1c_1}$ and the branches of hyperbola given by the relation $D = 0$ (see Fig. 13a).

3. $c_1 > c_2 > c_3 = 0$ (i.e., $c_1 > 0$ and $c_2 > 0$); in this case, the domains given by relations (5.9) are located between the lines $b_1 = \pm\sqrt{a_1c_1}$, $b_2 = \pm\sqrt{a_2c_2}$ and the branches of hyperbola $D = 0$ (see Fig. 13b).

Remark. It can be shown that the curves $\Phi = 0$ and $D = 0$ cross each other at the same points where they cross any line $b_i = \pm\sqrt{a_ic_i}$.

If $b_1 = b_2 = 0$, then conditions (5.9) lead to the conditions mentioned above [11]. Namely, only in case 3, where the axis corresponding to the maximal apparent additional mass is vertical, the solution turns out to be stable. Thus, the adding of the matrix **B** allows one to stabilize the motion (at least in the linear sense) for which the "middle" axis is vertical, and does not allow one to stabilize the motion for which the "small" axis is vertical.



(a) $c_1 = 1.1$, $c_2 = -0.1$ (b) $c_1 = 1.6$, $c_2 = 0.1$

Fig. 13. The typical pattern of domains on the plane of parameters b_1, b_2 (indicated by the gray color) for which the necessary conditions of stability (5.9) of Steklov solutions hold under different relations between parameters of matrix **C**. Here, $\mathbf{A} = \mathrm{diag}(1, 1.2, 2)$.

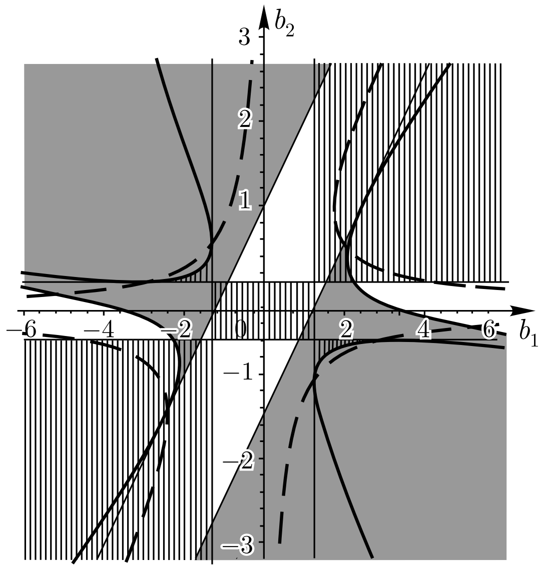


Fig. 14. The typical pattern of domains of stability on the plane of parameters b_1, b_2 of Steklov solutions corresponding to the fall by the wide (i.e., the eigenvector in the direction of the maximal added mass is vertical) and the middle side downward. $\mathbf{A} = \mathrm{diag}(1, 1.2, 2)$, $\mathbf{C} = \mathrm{diag}(1.6, 0.1, 0)$.

Now we put $c_1 > c_2 > c_3 = 0$ for definiteness, and plot on the plane of parameters b_1, b_2 the domains of (linear) stability of the Steklov solutions corresponding to the fall by the wide and middle side; see Fig. 14 (the fall by the narrow side is always unstable). It is well seen from the figure that there are domains where all three Steklov solutions are unstable (they are indicated in white).

5.2. Lyapunov stability. For one of the Steklov solutions (5.4), namely, for the case when the body falls by the "wide" side downward, one can prove the asymptotic Lyapunov stability.

As mentioned above, without loss of generality, we can put $i = 1$, $j = 2$, $k = 3$ and $b_3 = 0$, $c_1 > c_2 > c_3 = 0$ in (5.4). We construct the Lyapunov function in the form

$$V = H_2 + \frac{1}{\tau}W,$$

where H_2 is the quadratic part of the Hamiltonian near this solution in variables γ_1, γ_2, v_1, v_2:

$$H_2 = \frac{1}{2}(a_2^{-1}v_1^2 + a_1^{-1}v_2^2) + \frac{1}{2a_1}\Big(a_1c_1 - b_1^2 - \frac{2\sigma}{\sqrt{2\tau}}a_3b_1 + \frac{\sigma^2}{4\tau^2}a_3(a_1 - a_3)\Big)\gamma_1^2$$
$$+ \frac{1}{2a_2}\Big(a_2c_2 - b_2^2 - \frac{2\sigma}{\sqrt{2\tau}}a_3b_2 + \frac{\sigma^2}{4\tau^2}a_3(a_2 - a_3)\Big)\gamma_2^2, \tag{5.10}$$

and we look for the function W in the form of a homogeneous quadratic form in γ_1, γ_2, v_1, v_2 with constant coefficients.

It is easy to see that for large τ, the function H_2, and hence, V, is positive definite near the origin under the conditions

$$a_1c_1 - b_1^2 > 0, \quad a_2c_2 - b_2^2 > 0. \tag{5.11}$$

As shown above, these inequalities give us one of the domains of stability for the solution under consideration in the linear approximation (see Fig. 13b). Thus, we can show the asymptotic stability in the domain bounded by inequalities (5.11) for those values of parameters for which we will be able to choose a function V whose derivative along the solutions of a linear system is strictly negative (for sufficiently large τ).

The derivative of function V along solutions of system (5.6) has the form

$$\frac{dV}{d\tau} = -\frac{1}{\tau}G_1 + \frac{1}{\tau^{3/2}}G_2 + \frac{1}{\tau^2}G_3,$$

where G_1, G_2, G_3 are homogeneous quadratic forms in variables γ_1, γ_2, v_1, v_2. Thus, for large τ, the sign of the derivative $\dfrac{dV}{d\tau}$ is determined by the quadratic form G_1 which must be positive definite in the case of the asymptotic stability.

By the straightforward calculations it can be shown that it is necessary to choose W in the form

$$W = k_1v_1\gamma_1 + k_2v_2\gamma_2;$$

then G_2 and G_3 are independent of v_1, v_2 and

$$G_1 = 2k_1a_1^{-1}a_2(a_1c_1 - b_1^2)v_1^2 + 2k_2a_2^{-1}a_1(a_2c_2 - b_2^2)v_2^2 + a_2^{-1}(1 - 2a_2k_1)\gamma_1^2 + a_1^{-1}(1 - 2a_1k_2)\gamma_2^2$$
$$+ \frac{1}{2}a_1^{-1}\big(b_1 - 2k_1(a_1b_2 + a_2b_1)\big)\gamma_1v_2 - \frac{1}{2}a_2^{-1}\big(b_2 - 2k_2(a_1b_2 + a_2b_1)\big)\gamma_2v_1. \tag{5.12}$$

It is easy to obtain the conditions for positive definiteness of the form G_1:

$$0 < k_1 < \frac{1}{2a_2}, \quad 0 < k_2 < \frac{1}{2a_1},$$
$$-4(a_1b_2 + a_2b_1)^2k_1^2 - 16a_1a_2(c_1a_1 - b_1^2)k_1k_2 - b_1^2 + 4(2a_1a_2c_1 - a_2b_1^2 + a_1b_1b_2)k_1 > 0, \tag{5.13}$$
$$-4(a_1b_2 + a_2b_1)^2k_2^2 - 16a_1a_2(c_2a_2 - b_2^2)k_1k_2 - b_2^2 + 4(2a_1a_2c_2 - a_1b_2^2 + a_2b_1b_2)k_2 > 0.$$

There are two cases:

1. $b_1 \cdot b_2 > 0$, then, choosing $k_1 = \frac{1}{2}b_1(a_1b_2 + a_2b_1)^{-1}$, $k_2 = \frac{1}{2}b_2(a_1b_2 + a_2b_1)^{-1}$, we obtain the diagonal quadratic form (5.12) which is obviously positive definite;

2. $b_1 \cdot b_2 < 0$, in this case the sufficient solvability conditions for inequalities (5.13) are determined by solutions of a quartic equation (and have a rather inconvenient form). At the same time, since only one term is positive in the last two relations (5.13), we can obtain the necessary conditions for the solvability of (5.13) in the form

$$\Phi_1 = 2a_1a_2c_1 - a_2b_1^2 + a_1b_1b_2 > 0, \quad \Phi_2 = 2a_1a_2c_2 - a_1b_2^2 + a_2b_1b_2 > 0,$$
$$b_1b_2 > \max(-a_1c_2, -a_2c_1). \tag{5.14}$$

In Fig. 15 by the gray color the domain is shown where the necessary conditions (5.14) hold. As can be seen from the figure, for $b_1b_2 < 0$ the domain of asymptotic stability does not coincide with the whole domain of fixed sign of quadratic form (5.10).

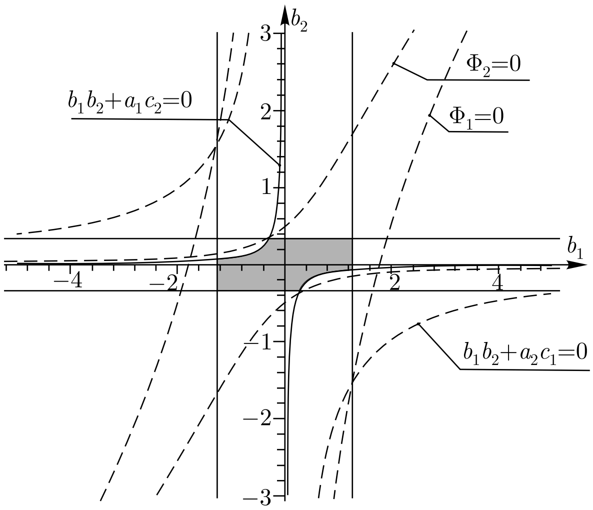


Fig. 15. The domain of asymptotic stability of the Steklov solution corresponding to the fall by the wide side downward with $\mathbf{A} = \mathrm{diag}(1, 1.2, 2)$, $\mathbf{C} = \mathrm{diag}(1.6, 0.1, 0)$.

Remark. An analysis of the (linear and nonlinear) stability for Steklov solutions was carried out in [6, 24]. In particular, conditions (5.13) were obtained in the form of general inequalities for coefficients without taking into consideration those Steklov solutions for which the relation between stability and instability can be different (see above). Here we carried out a geometric analysis of values of possible parameters for which the conditions of stability (5.13) hold, and drew the conclusion about the existence of a domain of values for the parameters for which all Steklov solutions are unstable. In this case, in the phase space there exists a more complex invariant attracting set of two-dimensional torus type (see Fig. 12) to which the trajectories of system (5.1) tend as $t \to +\infty$. Analytically, the existence of this invariant set remains unproven, because for the present bifurcation theory and qualitative methods have not been developed for systems of type (5.1) for which the linear "dissipation" decreases with time with respect to values of the parameter $\varepsilon \sim \dfrac{1}{\tau}$. In our analysis, simpler conditions for linear stability and Lyapunov stability are also obtained due to the systematic use of the Hamiltonian form of equations of motion.

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