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Euler and mathematical methods in mechanics (on the 300th anniversary of the birth of Leonhard Euler)

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Abstract. This article concerns the life of Leonhard Euler and his achievements in theoretical mechanics. A number of topics are discussed related to the development of Euler's ideas and methods: divergent series and asymptotics of solutions of non-linear differential equations; the hydrodynamics of a perfect fluid and Hamiltonian systems; vortex theory for systems on Lie groups with left-invariant kinetic energy; energy criteria of stability; Euler's problem of two gravitating centres in curved spaces.

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§ 1. Introduction

This article is devoted to the creative work of the brilliant mathematician Leonhard Euler. His heritage in mathematics and its applications is so great that it has not yet been fully appreciated and systematized. Euler's life is a striking example of the international character of science (and mathematics in particular). He was born and educated in Switzerland. There he revealed his mathematical talent under the guidance of the famous Johann Bernoulli, who in turn had learned directly from the great Leibniz himself. When he was 19 years old, Euler came to St. Petersburg on the advice of his friends Nikolaus and Daniel Bernoulli, in order to get a stable and well-paid position. As remarked by Condorcet, "they put as much effort into bringing their old competitor to them as ordinary men would have done to distance themselves from a rival." And indeed, we shall see later that

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Leonhard Euler and Daniel Bernoulli often competed, working at the same time on the same problems in theoretical mechanics.

Nowadays, young Russian mathematicians often go to Europe and America, with the same goals as Euler. In Euler's time everything was quite the other way round. Of course, the young city Petersburg then bore little resemblance to a European metropolis. Euler and Daniel Bernoulli, who at the time was working in Russia, were in fact of about the same age as the city. Daniel complained in his letters to his father Johann about the cold and boredom in Petersburg. Though he agreed, Johann responded that nevertheless there was no longer a place in Europe where science was so esteemed and well-paid.

It is interesting to know that Euler was first appointed an adjunct in the Physiology Department, there being no vacancies in physics and mathematics. He even made a serious attempt to learn some anatomy and medicine in his native city of Basel before coming to Petersburg. However, he was soon transferred, first to the Physics Department, and then in 1733 to the position previously occupied by his friend Daniel Bernoulli in the Mathematics Department. Daniel returned to Basel because of health problems; his older brother Nikolaus had died a year after arriving in Petersburg. Incidentally, the inscription under the bust of Euler in the main building of the Russian Academy of Sciences continues to maintain that Leonhard Euler was a physiologist, physicist, and (only then!) mathematician.

Overall, Euler worked in Petersburg for about 31 years and died here at the age of 76. When in about 1740 the general situation in Russia became less favourable for science, Euler accepted an invitation from the Prussian Emperor Frederick II, and he worked as a mathematics class director at the Berlin Academy for 25 years. It should be noted that, while in Berlin, Euler continued actively collaborating with the Petersburg Academy, regularly publishing his many papers on mathematics and mechanics in the Academy's *Commentarii*. Even after his death the journals of the Petersburg Academy of Sciences were still full of his treatises for many years.

Euler was the most authoritative and respected scientist in Russia. One may recall the well-known occasion when Princess Dashkova, who had been appointed director of the Petersburg Academy, asked Euler in particular to accompany her on her first visit to the academy (although at the time he did not get along with the head of the academy and did not even attend academy conferences).

It is said that Euler once decided to ask Empress Catherine II to give him the rank of general in the Privy Council (as an academician, he was a Counselor of State, which corresponded to the army rank of colonel). An example for Euler was Christian Goldbach, who held a government position and worked as a cryptographer in the Ministry of Foreign Affairs in Moscow. (Euler carried on an active correspondence with Goldbach in which they discussed, in particular, conjectures in additive number theory that had become famous.) However, Catherine joked: "Mister Academician, I have many generals but only one Euler!"

§ 2. Divergent series and a converse of the Lagrange–Dirichlet theorem

There is the opinion (and not just in works on the history of science) that mathematicians of that time (Euler in particular) often displayed muddled thinking in elementary questions. For example, they did not distinguish between differentiable and continuous functions, mixed up convergent and divergent series, and so on. It is true that there was then less rigorous scholarship, it was not common to formulate definitions with precision, and many mathematical structures were not singled out and clearly represented. However, one should be cautioned against taking a superficial view in this matter. It would be very naive to think that Euler did not understand the difference between divergent and convergent series. Hardy rightly observed concerning these questions that it was rather a lack of techniques then available to mathematicians. The Cauchy convergence criterion was not yet known, so it was difficult to make a conclusion about convergence of a series without trying to calculate its sum. When dealing with divergent series, Euler's approach was much more flexible and also deeper than that in a standard university calculus course. When a university graduate who has mastered such a course encounters a divergent series, he disregards it as something forbidden or even non-existent. But it might be useful to try to find the sum of the divergent series by applying an appropriate method of summation. This problem can frequently be resolved as follows. As we know, the bulk of ergodic theory is based in essence on replacing ordinary convergence by Cesàro convergence (arithmetic means).

Here is one of Euler's results:

$$0! - 1! + 2! - 3! + 4! - \cdots = 0.5963\dots\,. \tag{2.1}$$

He obtained it by several different methods.

However, there is the opinion of Academician A. N. Krylov:

> All such formulae in mathematics are completely useless...

The followers of Euler, in bowing before his authority, seemed later to try to invent even more such absurd equalities, often forgetting about the conditional meaning attributed to them by Euler, and thereby created *that scandal in mathematics* which lasted for 75 years...

The scandal was ended by Gauss, Cauchy, and Abel, who banished from rigorous mathematics the use of series whose convergence had not been investigated [1].

This conservative point of view is obviously not very productive. By the time Krylov wrote these lines, the theory of summation of divergent series was already quite advanced (see the comprehensive monograph of Hardy [2], who treated the history of this question at length, in particular, Euler's contribution to the development of the theory).

Here is the way to look at Euler's formula (2.1). Consider the following simple non-linear system of differential equations in the plane $\mathbb{R}^2 = \{x, y\}$:

$$\dot{x} = x - y, \qquad \dot{y} = -y^2. \tag{2.2}$$

Let us look for solutions asymptotically approaching the equilibrium point $x = y = 0$ as $t \to +\infty$. The second equation in (2.2) has the obvious asymptotic solution $y(t) = 1/t$. Substituting it into the first equation, one finds a formal solution of the resulting non-homogeneous linear equation in the form of a series of negative powers of t:

$$x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k-1)!}{t^k}\,. \tag{2.3}$$

For $t = 1$ one gets Euler's series. On the other hand, the system (2.2) has a solution

$$x(t) = e^t \int_t^\infty \frac{e^{-u}}{u}\, du, \qquad y(t) = \frac{1}{t}$$

tending to zero, and the divergent series (2.3) is an asymptotic representation of the function $x(t)$ as $t \to +\infty$ (it is easy to see this by successive integration by parts). For $t = 1$ one gets Euler's sum of the series:

$$e \int_1^\infty \frac{e^{-u}}{u}\, du = 0.5963\ldots.$$

This argument differs only in form from the argument of Euler himself.

In general, suppose that the system of differential equations

$$\dot{x} = v(x), \qquad v(0) = 0, \tag{2.4}$$

in $\mathbb{R}^n = \{x\}$ with smooth (infinitely differentiable) right-hand side admits a formal solution in the series form

$$\sum_{j=1}^{\infty} \frac{x^{(j)}}{(t^\mu)^j}, \qquad x^{(j)} \in \mathbb{R}^n, \tag{2.5}$$

where μ is some positive constant ($\mu = 1$ for the solution (2.3)). Then this equation has a true solution $t \mapsto x(t)$ (and maybe more than one) such that

1) $x(t) \to 0$ as $t \to \infty$,
2) the power series (2.5) will be asymptotic for it, that is,

$$x(t) - \sum_{j=1}^{N} \frac{x^{(j)}}{(t^\mu)^j} = O\biggl(\frac{1}{t^{(N+1)j}}\biggr), \qquad t \to \infty.$$

This remarkable theorem was proved by A. N. Kuznetsov in 1972 [3]. I think that Euler himself would be pleased with this result. Kuznetsov later (at my suggestion) generalized the theorem to the case when the coefficients of the formal series (2.5) depend polynomially on the logarithm of t. He also extended the theorem to more general scales of comparison [4].

Unfortunately (and surprisingly), it must be said that Kuznetsov's theory is hardly known (especially abroad). For example, in a recent book by J.-P. Ramis [5] devoted to asymptotic analysis of solutions of differential equations, Kuznetsov's theory is not even mentioned. Instead, an assertion is given that is weaker, and moreover was obtained later.

Let me give an example of ideas related to the problem of a converse of the Lagrange–Dirichlet stability theorem, a problem posed by A. M. Lyapunov as far back as 1897. The Lagrange–Dirichlet theorem states that if at an equilibrium state the potential energy of a system has a strict local minimum, then the equilibrium is stable. This *energy criterion for stability* is widely used also in studying the stability of equilibrium states for systems with continuously distributed parameters. In this connection it is important to find necessary conditions for stability.

The motion of a system in a potential force field is described by the Lagrange equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = -\frac{\partial V}{\partial x}, \qquad x \in \mathbb{R}^n, \tag{2.6}$$

where

$$T = \frac{1}{2}\sum g_{ij}(x)\dot{x}_i\dot{x}_j$$

is a positive-definite quadratic form (the kinetic energy of the system) in the velocities $(\dot{x}_1, \dots, \dot{x}_n) = \dot{x}$, and $V\colon \mathbb{R}^n \to \mathbb{R}$ is the potential energy. Let $dV(0) = 0$. Then the point $x = 0$ is an equilibrium state. Suppose that the functions g_{ij} and V are infinitely differentiable, and

$$V = V(0) + V_2 + V_m + V_{m+1} + \cdots \tag{2.7}$$

is the Maclaurin expansion of the potential energy in a series of homogeneous forms $(V_k(\lambda x) = \lambda^k V_k(x))$. Since $x = 0$ is an equilibrium state, the expansion does not contain terms linear in $x_1, \dots, x_n$.

Of course, in a typical situation $m = 3$. However, for even potentials $m = 4$.

As shown by Lyapunov, if V_2 does not have a minimum at the point $x = 0$, then this equilibrium is unstable (exponentially so). That is why we consider the case when $V_2 \geqslant 0$. On the other hand, if the quadratic form V_2 is positive-definite, then the equilibrium at $x = 0$ is stable (the Lagrange–Dirichlet theorem). Thus, it is useful to consider the plane

$$\Pi = \{x \in \mathbb{R}^n : V_2(x) = 0\}$$

and to assume that $\dim \Pi \geqslant 1$. Let W_m be the restriction of the homogeneous form V_m to Π. Then the following theorem holds.

Theorem 2.1 [6]. *If W_m does not have a minimum at the point $x = 0$, then the equilibrium is unstable.*

In this case there is a formal solution of (2.6) as the series (2.5) with $\mu = 2/(m-2)$. For even $m \geqslant 4$, the coefficients $x^{(j)}$ still depend polynomially on $\log t$. If $V_2 \not\equiv 0$ then the series (2.5) diverge as a rule, even under the assumption that the functions g_{ij} and V are analytic. However (according to Kuznetsov's theorem [4]), (2.6) has a solution $t \mapsto x(t)$ which tends to the equilibrium state $x = 0$ as $t \to +\infty$. Since the equation (2.6) does not change under time reversal $t \mapsto -t$, it follows that $t \mapsto x(-t)$ will be a solution *asymptotically going out* from the point $x = 0$. In particular, the equilibrium at $x = 0$ is unstable, and this instability is of power order.

It should be noted that Theorem 2.1 was published about three years before Kuznetsov's general theorem, which is necessary for proving it. In fact, Kuznetsov's theorem with logarithms was announced in the paper [6].

If the expansion (2.7) does not contain quadratic additive terms ($V_2 = 0$), then in the analytic case the formal series (2.5) converges for sufficiently large values of t [7]. This was established even in the more general case when x is a point in a Hilbert space. If $V_2 \not\equiv 0$ and V_2 does not have a minimum at the point $x = 0$, then the equilibrium of the infinite-dimensional system is also unstable. Here one should look for an asymptotic solution in the form of a series of powers of $\exp(-\lambda t)$, $\lambda > 0$, which converges for large t. These results (along with the Lagrange–Dirichlet theorem) give a sound basis for investigating the stability of equilibrium states of a system with continuously distributed parameters. It would interesting to extend Theorem 2.1 to the infinite-dimensional case. This would give the possibility of proving the instability of an equilibrium at bifurcation points when the second differential of the potential energy loses positive definiteness.

A systematic presentation of the above circle of questions can be found in the book [8].

§ 3. Euler and mechanics

Let us now turn to Euler's work devoted specifically to mechanics. This topic seems difficult to cover, especially in view of his numerous investigations of applications. Therefore, we focus on his purely theoretical work on the mathematical aspects of mechanics.

We first discuss Euler's achievements in analytical hydrodynamics.

A. Euler introduced and systematically used the so-called *natural* equations of motion of a material point, when Newton's equation $m\ddot{r} = F$ is projected on the axes of the moving Frenét frame. He later used this idea in the dynamics of rigid bodies. It is interesting to note that a simpler form of the equations of motion in a fixed reference frame appeared later in works of Maclaurin.

Euler discovered that in the absence of external forces the trajectories of a material particle moving on a smooth regular surface coincide with geodesics of the surface.

Euler (simultaneously with D. Bernoulli) introduced into mechanics the concept of angular momentum of the motion of a system of particles and found the fundamental theorem on change of angular momentum. The theorem employs the concept of moment of force or torque as well, which was known before Euler and widely used in problems in statics.

This theorem enabled Euler to derive the equations of a rotating rigid body:

$$I\dot{\omega} + \omega \times I\omega = M. \tag{3.1}$$

Here ω is the angular velocity of the body in a moving reference frame (attached to the body), I is the operator (tensor) of inertia, and M is the total moment of the forces acting on the body. In deriving this equation Euler used his theorem on the velocity distribution in a moving rigid body along with the theory of moments of inertia.

When he was studying the change in the moment of inertia of a body in dependence on the direction of a line, Euler proved the existence of three mutually orthogonal lines (the principal axes of inertia) which have extremal values of the moments of inertia. In this reference frame the symmetric operator I is reducible to diagonal form. These results are essentially the first theorems of linear algebra on transformations of quadratic forms.

Euler introduced coordinates determining the orientation of the body (Euler's angles), and he represented the angular velocity in terms of these angles and their derivatives with respect to time (Euler's kinematic formulae). These relations together with the dynamical equation (3.1) form a closed system of differential equations describing the rotation of a rigid body about a fixed point under an action of given forces. Euler integrated his equations in the simple but important case when $M = 0$ (the Euler top).

There is no doubt that Euler was the founder of the dynamics of a rigid body, which besides all else has had great value in applications.

Euler formulated and justified the principle of least action. This variational principle was first established for the motion of a single material point in a potential force field, then Lagrange extended it to a system of interacting particles, and Jacobi later expressed it in the form familiar to us. It is interesting to note that Euler and Lagrange, the classicists of variational calculus, failed to formulate the simpler and more fundamental variational principle known nowadays as Hamilton's principle, from which the principle of least action can be deduced as a corollary. Variational principles play an essential role not only in analytical mechanics but also in mathematical and theoretical physics.

The history of the discovery of the principle of least action is quite instructive. I shall present it in Jacobi's very vivid account (somewhat abridged) [9].

> The origin of this principle was connected with a great controversy in the literature, such an uproar as had hardly happened before. The discovery of the principle is attributed to a former president of the Berlin Academy, Maupertuis. Though he was an intelligent man with some good ideas, he had not learned much and did not understand the recently discovered calculus of infinitesimals as well as the great mathematicians of that time…
>
> …in 1744 Maupertuis arranged his report to the Paris Academy of Sciences with great pomp. In the report he stated that mathematics thus far had been used everywhere for the most insignificant purposes, but now he would find worthy applications of it, deriving all of Nature's laws directly from God's attributes. Under the name 'the principle of least action' he formulated a principle for which no dog would leave his warm spot by the stove, whereas Euler's principle encompasses the whole of mechanics…
>
> Maupertuis made public his theorem not long before departing from Paris to Berlin to become the president of the Academy there, and then he wrote a second treatise about the theorem in the Berlin *Mémoiren*. The tone of the treatise soon met with natural objections…
>
> …a mathematician named König was especially indignant. He was a strange man: in his youth he, together with the same Maupertuis and also Clariaut, had studied under the old Johann Bernoulli in Basel, but he was later involved in demagogic agitation… directed against the aristocracy, and as a result was exiled from his native city of Bern…
>
> It was he who attacked Maupertuis in *Nova Acta Eruditorum*, a very well-known journal… In his article König included an excerpt from a letter of Leibniz…, in which Leibniz made clear that he had

> carefully studied the concept of action, and that the action would always be minimal...
>
> Consequently, a huge scandal erupted, and König was requested to produce the original letter. However, he had only a copy given him by a certain Henzi, who had been hanged during the unrest in Zürich mentioned above... Later, König published the letter in its entirety, and it did contain much of interest, but it seemed that he had made up the excerpt about the action being minimal, just in order to irritate Maupertuis. Euler was asked to write a report exposing König, and the Berlin Academy solemnly confirmed König's deception at one of its meetings...
>
> ...Euler always substituted his own principle of least action for the Maupertuis principle and gave as a reason that he identified the two principles. He said that if the principle he had proved for particular cases was so important, then what should one say about the principle of Maupertuis, whom he called the discoverer of all the laws of mechanics and physics.

B. Leonhard Euler was one of creators of hydrodynamics. The equations of motion of a fluid can be represented in two different forms: we can study the velocity, pressure, and density at all points of the flow, or we can study the history of every single particle of the fluid. The equations obtained in these two ways are usually called Euler's and Lagrange's forms of the equations of hydrodynamics (though both are due to Euler).

Euler's equations

$$\rho\left(\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}v\right)=-\frac{\partial p}{\partial x}-\rho\frac{\partial V}{\partial x} \tag{3.2}$$

describing the motion of a perfect fluid are the most widely used. Here $x = (x_1, x_2, x_3)$ is a point in three-dimensional Euclidean space, $v(x,t)$ is the velocity field of the particles of the fluid, $\rho(x,t)$ is the density, $p(x,t)$ is the pressure, and $V(x,t)$ is the potential energy density of the mass forces. Accompanying these three scalar equations (3.2) is the continuity equation

$$\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho v)=0, \tag{3.3}$$

which expresses the law of conservation of mass of the moving volume of fluid. The last equation was also obtained by Euler. One usually considers a barotropic fluid, when the pressure and density are connected through some functional relation. This additional assumption closes the system of non-linear differential equations (3.2) and (3.3). Euler's main results in hydrodynamics were published in 1755 (Berlin) and 1759 (St. Petersburg).

For a barotropic fluid the equation (3.2) can be represented in Lamb's form

$$\frac{\partial v}{\partial t}+(\operatorname{rot} v)\times v=-\frac{\partial h}{\partial x}\,, \tag{3.4}$$

where $h = v^2/2 + \mathscr{P} + V$, and $\mathscr{P}$ is a function of the pressure ($\rho^{-1}dp = d\mathscr{P}$). The function h is called the Bernoulli function. In the stationary case h is constant along stream lines (the Bernoulli theorem). This fact was basic in Daniel Bernoulli's *Hydrodynamica*, published in 1738.

Stationary flows of a perfect fluid have been extensively studied. In the first place, the vector fields v and $\operatorname{rot} v$ are tangent to the Bernoulli surfaces

$$B_c = \{x : h(x) = c\}.$$

These surfaces are regular if the field v is not collinear with its rotation. Moreover, the fields

$$v \quad \text{and} \quad (\operatorname{rot} v)/\rho$$

commute. This was noted by V. I. Arnol'd [10] for a homogeneous incompressible fluid ($\rho = \text{const}$) and by the author [11] in the general case. Therefore, if the Bernoulli surfaces are bounded, then they are diffeomorphic to two-dimensional tori, and the motion of the fluid particles is quasi-periodic. However, if the fields v and $\operatorname{rot} v$ are collinear, then the flow in a closed region will in general be chaotic.

C. In the mathematical theory of elasticity Euler derived the differential equation for an elastic curve (rubber band) and gave a classification of its equilibrium shapes (1744).

He determined the smallest height for which a column begins to bend under its own weight or an externally applied load. Lagrange followed Euler's theory and applied it to determine the safest shape for the column. These were the first studies on the stability of elastic systems, and the energy criterion for stability played a key role here (for systems with finitely many degrees of freedom, this was the Lagrange–Dirichlet theorem mentioned above).

Finally, after Taylor, Euler obtained the differential equation for transverse oscillations if a rod, found the eigenfunctions, and derived equations for the frequencies under a variety of boundary conditions. Daniel Bernoulli obtained these results at the same time as Euler.

D. Euler's works on celestial mechanics and astronomy were mainly focused on applications (for example, practical tables were compiled from his theory of the motion of the moon). However, there were results of Euler on the three-body problem which went into the 'treasure house' of celestial mechanics. First of all, he found the particular motions of three gravitating bodies that always stay on a single moving straight line. Further, he integrated the planar problem of two fixed centres when a moving particle is attracted by the two fixed points. If one of the attracting centres is shifted to infinity and its mass is increased in a coordinated way, then we obtain an important limiting integrable problem of Kepler for a homogeneous force field. This problem has essential significance in quantum mechanics, for a hydrogen atom in a homogeneous electric field. It is related to the Stark effect: the splitting of the spectral lines of atoms in an electric field.

Lagrange later proved integrability in the problem of two centres in space, and also generalized it to the case when a moving point is attracted (or repelled) by an elastic force directed towards the mid-point between the two centres.

Jacobi rightly characterized Euler's scientific style in his *Lectures on Dynamics*:

> In general, Euler's works have the great merit that he always considered all the possible cases in which problems could be solved completely by given methods and tools... As a rule, it enriches science whenever one manages to add some new example to his examples.

In connection with this opinion we mention the remarkable result of S. V. Bolotin on non-integrability of the planar problem of n fixed centres for $n \geqslant 3$ [12].

We now turn our attention to new studies in these areas where Euler's ideas and methods play a substantial role.

§ 4. Hydrodynamics of Hamiltonian systems

Euler's equations in hydrodynamics (3.2) arise naturally in the investigation of invariant manifolds of Hamiltonian systems that project one-to-one on the configuration space.

Let M^n be the n-dimensional configuration manifold for a mechanical system with local coordinates $x = (x_1, \dots, x_n)$, let $\Gamma = T^*M$ be the phase space (cotangent bundle of M), let $y = (y_1, \dots, y_n)$ be the canonical momenta at a point $x \in M$ (these are co-vectors, elements of the space T_x^*M dual to the tangent space T_xM), and let $H(x, y, t)$ be the Hamiltonian function (given on Γ and dependent in general on the time t). The evolution is determined by the system of canonical Hamilton equations

$$\dot{x}_j = \frac{\partial H}{\partial y_j}, \quad \dot{y}_j = -\frac{\partial H}{\partial x_j}, \qquad 1 \leqslant j \leqslant n. \tag{4.1}$$

Let Σ_t^n be a manifold in Γ that projects one-to-one on the configuration space M. In the canonical coordinates x, y it is given by equations

$$y = u(x, t), \tag{4.2}$$

where u is some co-vector field on M, possibly dependent on the time. The *invariance* property of the manifold Σ_t has the following meaning. Let $x(t)$, $y(t)$ be a solution of the Hamilton equations with the initial data

$$x(t_0) = x_0, \qquad y(t_0) = y_0.$$

If a point (x_0, y_0) belongs to Σ_{t_0}, then

$$(x(t), y(t)) \in \Sigma_t$$

for all values of t.

It is easy to check that a criterion for invariance of the manifold Σ_t is that the field (4.2) satisfies the equation

$$\frac{\partial u}{\partial t} + (\operatorname{rot} u)v = -\frac{\partial h}{\partial x}, \tag{4.3}$$

where $\operatorname{rot} u = \partial u/\partial x - (\partial u/\partial x)^T$ is a skew-symmetric $n \times n$ matrix (the rotation of the co-vector field u),

$$v(x, t) = \frac{\partial H}{\partial y}\bigg|_{y=u(x,t)}$$

is a vector field on M, and $h(x, t) = H(x, u(x, t), t)$ is a function on M depending on the time t as a parameter.

For the 'natural' Hamiltonian

$$H = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}(x,t)y_i y_j + V(x_1,\dots,x_n,t)$$

the equation (4.3) has the explicit form

$$\begin{aligned}\frac{\partial u_i}{\partial t} &+ \sum_{j,k} g_{jk}\biggl(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\biggr)u_k \\ &= -\frac{1}{2}\frac{\partial}{\partial x_i}\biggl(\sum_{j,k} g_{jk}u_j u_k\biggr) - \frac{\partial V}{\partial x_i}, \qquad i = 1,\dots,n.\end{aligned} \tag{4.4}$$

This is a non-linear system of n first-order partial differential equations with respect to the n unknown functions $u_1,\dots,u_n$.

For example, take $n = 3$ and $g_{ij} = \delta_{ij}$ (Kronecker symbol). Then $u = v$ and the equations (4.4) will have the form of Euler's equations in hydrodynamics, in Lamb's representation. The function h will be called the Bernoulli function.

The motions of this system with phase trajectories lying on Σ can be found as solutions to the following system of ordinary differential equations on M:

$$\dot{x} = v(x,t), \qquad x \in M. \tag{4.5}$$

These considerations allow the use of the ideas and methods of hydrodynamics in the study of Hamiltonian systems [13]. For this purpose, let us introduce *vortex manifolds* as a natural generalization of the vortex lines in classical hydrodynamics. The eigenvectors of the matrix $\operatorname{rot} u$ with zero eigenvalue will be called *vortex vectors*. At each point $x \in M$ and at any time t the tangent vortex vectors form a linear space, and a distribution of tangent planes on M arises. It can be shown (assuming that the planes of vortex vectors have constant dimension) that this distribution is integrable. It is natural to call the integral manifolds of the distribution *vortex manifolds*.

Vortex manifolds turn out to be 'frozen' into the flow of the system of differential equations (4.5) on M. In other words, the transformations determined by this flow carry vortex manifolds into vortex manifolds. This is a multidimensional variant of the classical Helmholtz–Thomson theorem in the hydrodynamics of a perfect fluid. Also, the following natural generalization of the Bernoulli theorem holds: in the stationary case the Bernoulli function h is constant on both stream lines (integral curves of the vector field v on M) and vortex manifolds.

Let us consider a potential solution of Lamb's equation (4.3) when $u = \partial S/\partial x$, where S is a function of the coordinates x and the time t. For this case the equation (4.3) is equivalent to the equation

$$\frac{\partial S}{\partial t} + H\biggl(x, \frac{\partial S}{\partial x}, t\biggr) = f(t). \tag{4.6}$$

The trivial gauge transformation

$$S \mapsto S - \int f(t)\,dt$$

leads to the vanishing of the right-hand side of (4.6), which becomes the well-known Hamilton–Jacobi equation. In hydrodynamics, (4.6) is called the Lagrange–Cauchy integral.

Jacobi's classical method of integrating the Hamilton differential equations (4.1) leads to the *complete* solution of (4.6), which is non-degenerately dependent on an additional n parameters $c_1, \dots, c_n$:

$$\det\left\| \frac{\partial^2 S}{\partial c_i\, \partial x_j} \right\| \neq 0.$$

A generalization of Jacobi's method based on finding the complete solution of Lamb's equation is indicated in [14]. Let me give the simplest variant of the integrability theorem in the stationary case with $n = 3$. Suppose that a solution $u(x, c)$, $c = (c_1, c_2, c_3)$, of (4.3) is known such that

1) $\operatorname{rot} u \neq 0$,
2) $\left| \dfrac{\partial u}{\partial c} \right| \neq 0$,
3) $d_x H(x, u(x, c)) \neq 0$.

Then the Hamilton equation can be integrated by quadratures.

This assertion can be proved with the help of the Euler–Jacobi 'last multiplier' theory. According to Jacobi, "...Euler's theory on the introduction of a multiplier in an integral equation is one of his most remarkable discoveries in integral calculus."

Apparently, the equation (4.3) first appeared in variational calculus as a *condition for the consistency of fields of extremals* (which are known to be described by the canonical Hamilton equations). It is true that there one usually considers only *self-conjugate* (potential) fields. I. S. Arzhanykh [15] generalized Lamb's equation to non-Hamiltonian (in particular, non-holonomic) systems and tried to extend the Hamilton–Jacobi method to this case. However, prior to the paper [13] the equation (4.3) was usually not connected with ideas in hydrodynamics. A systematic exposition of analogies among hydrodynamics, geometric optics, mechanics, and thermodynamics can be found in the book [16].

§ 5. Vortex theory of the Euler top

The key problem in applicability of the general vortex theory in § 4 is to find invariant manifolds that project one-to-one on the configuration space. This problem can be easily and naturally resolved for the Euler top [17].

Let α, β, γ be an orthonormal reference frame in the fixed space. We shall consider the basis components as vectors in a moving space connected with the rigid body. Then they are no longer constant; their evolution in time is described by the Poisson equations

$$\dot{\alpha} + \omega \times \alpha = 0, \qquad \dot{\beta} + \omega \times \beta = 0, \qquad \dot{\gamma} + \omega \times \gamma = 0. \tag{5.1}$$

These equations together with the Euler dynamical equations (3.1) constitute a complete system of equations of motion (under the assumption that the moment M of the forces acting on the body is a known function of the position and angular velocity of the rigid body). In the case $M = 0$ these equations admit the integrals

$$(I\omega, \alpha) = c_1, \qquad (I\omega, \beta) = c_2, \qquad (I\omega, \gamma) = c_3, \tag{5.2}$$

which express the constancy of the angular momentum vector $K = I\omega$ of the body as a vector in the fixed space.

The integrals (5.2) have a clear group theory interpretation. The configuration space of a top rotating about a fixed point is the group SO(3) of rotations of three-dimensional Euclidean space. The *right-invariant* vector field of velocities on SO(3) corresponds to the rotations of the top with constant angular velocity $\omega = \alpha$. The phase flow of this field obviously consists of the *left* translations on the group SO(3). However, the kinetic energy $T = (I\omega, \omega)/2$ of the top is invariant under left translations. Consequently, by Noether's theorem the equations of motion admit the integral

$$\left(\frac{\partial T}{\partial \omega}, \alpha\right) = (I\omega, \alpha) = \text{const.}$$

These considerations are actually contained in a brief note of Poincaré [18]. In the same note, more general systems on Lie groups with a left-invariant metric (the kinetic energy) are introduced; they are discussed below.

The formulae (5.2) obviously imply that

$$I\omega = c_1\alpha + c_2\beta + c_3\gamma. \tag{5.3}$$

This equality lets us represent the rotational velocity of the top as a single-valued function on the configuration space. In other words, the vector equality (5.3) determines a three-parameter family of three-dimension stationary invariant manifolds that project one-to-one on the configuration space $M = \mathrm{SO}(3)$. The non-trivial case when $c_1^2 + c_2^2 + c_3^2 \neq 0$ is considered below.

To simplify the formulae we associate with the rigid body the trihedron formed by its principle axes of inertia with respect the point that is fixed. In this coordinate system the tensor of inertia is reduced to diagonal form: $I = \mathrm{diag}(I_1, I_2, I_3)$. Let ω_1, ω_2, ω_3 be the projections of the angular velocity vector ω on these moving axes. Then the kinetic energy formula takes the simplified form

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2). \tag{5.4}$$

To represent the invariant manifolds (5.2) in the canonical variables, we introduce the well-known *Euler's angles* ϑ, φ, ψ as generalized coordinates on the group SO(3). They uniquely determine the position of the principle axes of inertia of the rigid body with respect to the fixed trihedron. The angular velocities can be expressed in terms of Euler's angles and their derivatives with the help of *Euler's kinematic formulae* (1760):

$$\begin{aligned}
\omega_1 &= \dot\psi \sin\vartheta \sin\varphi + \dot\vartheta \cos\varphi,\\
\omega_2 &= \dot\psi \sin\vartheta \cos\varphi - \dot\vartheta \sin\varphi,\\
\omega_3 &= \dot\psi \cos\vartheta + \dot\varphi.
\end{aligned} \tag{5.5}$$

They enable one to represent the kinetic energy (5.4) as a positive-definite quadratic form in the generalized velocities $\dot\vartheta$, $\dot\varphi$, $\dot\psi$. The conjugate canonical momenta are introduced according to the usual rule:

$$p_\psi = \frac{\partial T}{\partial \dot\psi}, \qquad p_\vartheta = \frac{\partial T}{\partial \dot\vartheta}, \qquad p_\varphi = \frac{\partial T}{\partial \dot\varphi}.$$

Using (5.4) and (5.5), one obtains the formulae

$$\begin{aligned}
p_\psi &= I_1\omega_1\frac{\partial\omega_1}{\partial\dot\psi} + I_2\omega_2\frac{\partial\omega_2}{\partial\dot\psi} + I_3\omega_3\frac{\partial\omega_3}{\partial\dot\psi} \\
&= I_1\omega_1\sin\vartheta\sin\varphi + I_2\omega_2\sin\vartheta\cos\varphi + I_3\omega_3\cos\vartheta, \\
p_\vartheta &= I_1\omega_1\cos\varphi - I_2\omega_2\sin\varphi, \\
p_\varphi &= I_3\omega_3.
\end{aligned}$$

Taking into account the formulae (5.3) and the well-known expressions for the components of the vectors α, β, γ in terms of Euler's angles, one obtains the definitive form of the invariant manifolds for the Euler top in the canonical variables:

$$\begin{aligned}
p_\psi &= c_3, \\
p_\vartheta &= c_1\cos\psi + c_2\sin\psi, \\
p_\varphi &= c_1\sin\vartheta\sin\psi - c_2\sin\vartheta\cos\psi + c_3\cos\vartheta.
\end{aligned} \tag{5.6}$$

It is interesting to note that these formulae do not depend on the moments of inertia I_1, I_2, I_3 of the body.

Since no forces are acting on the top, the fixed frame α, β, γ can be chosen arbitrarily. For example, the vector γ can be directed along the vector of angular momentum $K = I\omega$, which is constant in the space. Then it is clear that

$$c_1 = c_2 = 0, \qquad c_3 = k, \qquad k = |K|.$$

For a fixed k the invariant manifolds (5.6) determine on the group SO(3) a dynamical system

$$\dot x = v(x), \qquad x = (\psi, \vartheta, \varphi).$$

Using Euler's kinematic formulae, one can easily write these equations in the explicit form

$$\begin{aligned}
\dot\psi &= k\left(\frac{\sin^2\varphi}{I_1} + \frac{\cos^2\varphi}{I_2}\right), \\
\dot\vartheta &= k\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\sin\vartheta\sin\varphi\cos\varphi, \\
\dot\varphi &= k\cos\vartheta\left(\frac{1}{I_3} - \frac{\sin^2\varphi}{I_1} - \frac{\cos^2\varphi}{I_2}\right).
\end{aligned} \tag{5.7}$$

Their phase flow determines a stationary 'flow' of an imaginary fluid on the group SO(3). We describe its main properties.

1°. The 'intrinsic' metric on SO(3) given by the kinetic energy of the top enables us to compute the rotation of the vector field v (the field (5.7)). This is the vorticity field. Considered as a velocity field, it generates a rotation of the rigid body with angular velocity $\omega = \mu\gamma$, $\mu = \text{const}$, and it commutes with the field v. In particular, the field $\operatorname{rot} v$ is right invariant, and all the vortex lines are closed. The fibring of the group SO(3) by the vortex lines coincides with the *Hopf bundle* well known in topology.

2°. The equations (5.7) admit the integral invariant

$$\iiint \sin\vartheta\, d\psi\, d\vartheta\, d\varphi, \qquad 0 \leqslant \vartheta \leqslant \pi.$$

It determines a measure on SO(3) that is invariant with respect to left and right translations.

This fact allows one to speak about the conservation of the 'mass' of the imaginary fluid on the group SO(3). The mass density coincides with the density of the bi-invariant measure (which is unique up to a constant factor).

3°. The Bernoulli integral h equals

$$k^2(I^{-1}\gamma,\gamma)/2.$$

The critical points of the function h are the orbits of constant rotations of the top about the principal axes of inertia (with fixed value of the angular momentum K), and the critical values coincide with the values of the energy for these rotations. If c is not a critical value of h, then the Bernoulli surface

$$B_c = \{x \in \mathrm{SO}(3) : h(x) = c\}$$

is a two-dimensional torus. The vector fields v and $\operatorname{rot} v$ are tangent to B_c, commute, and are linearly independent at each point. In particular, the motion on this torus is quasi-periodic.

These results can be carried over (after natural modifications) to the general case of systems on Lie groups with a left-invariant kinetic energy. Let G be a connected Lie group, the configuration space of a mechanical system, and let $T(\dot{x}, x)$ be the kinetic energy, which is invariant with respect to all left translations on G.

Let $w_1, \dots, w_n$ be a basis of right-invariant vector fields on G. Their phase flows are families of left translations on G. Thus (by Noether's theorem), the equations of motion admit n independent first integrals

$$\frac{\partial T}{\partial \dot{x}} \cdot w_1 = c_1, \qquad \dots, \qquad \frac{\partial T}{\partial \dot{x}} \cdot w_n = c_n,$$

which are linear with respect to the velocity $\dot{x}$. From these relations one can express the velocity of the system as a single-valued function of the position x and the parameters $c = (c_1, \dots, c_n)$:

$$\dot{x} = v_c(x), \qquad x \in G. \tag{5.8}$$

Let us recall that on each Lie group there is a unique (up to a constant factor) measure that is invariant under all left (right) translations. In the case of a *unimodular* group this measure (called the Haar measure) is bi-invariant. In particular, all compact groups are unimodular.

Theorem 5.1 [19]. *For a fixed value of $c \in \mathbb{R}^n$, the phase flow of the system (5.8) preserves a right-invariant measure on G.*

Corollary. *If the group G is unimodular, then the phase flow of the system (5.8) preserves the Haar measure on G.*

This statement is a generalization of the property 2° of the Euler top.

This approach can also be applied to the infinite-dimensional group of diffeomorphisms of a smooth manifold Q that preserve the volume element. To connect the constructions below with the usual hydrodynamics of a perfect fluid (say, with periodic boundary conditions), we can take Q to be, for example, a three-dimensional torus with a flat metric and with angular coordinates $x = (x_1, x_2, x_3)$. This group is usually denoted by $\mathrm{SDiff}\, Q$. The Lie algebra of $\mathrm{SDiff}\, Q$ consists of the tangent vector fields on Q with zero divergence. Let us define the inner product of two elements of this algebra (that is, two solenoidal vector fields v_1 and v_2) by the formula

$$\langle v_1, v_2 \rangle = \int (v_1, v_2)\, d^3x.$$

We now consider the flow of a homogeneous perfect fluid in the region Q, with the density of the fluid taken to be 1 for simplicity. The continuity equation gives the condition of incompressability: $\operatorname{div} v = 0$. The fluid flow is described by curves g^t on the group $\mathrm{SDiff}\, Q$: the diffeomorphism $g^t \colon Q \to Q$ carries each particle from its initial position to its position at time t.

It is easy to check that the kinetic energy

$$T = \frac{1}{2} \langle v, v \rangle \tag{5.9}$$

of the fluid is a *right-invariant* Riemannian metric on the group $\mathrm{SDiff}\, Q$. The property of right invariance distinguishes this system from the Euler top and creates additional difficulties in the investigation of the system.

In the 1960s the following important circumstance was noted: flows of a perfect incompressible fluid are geodesics of the metric (5.9). Thus, a perfect incompressible fluid is an infinite dimensional 'Euler top' with a right-invariant metric on the group $\mathrm{SDiff}\, Q$ ([20]–[22]). This result is a consequence of the principle of least action, which can if desired be regarded as the definition of the dynamics of a perfect fluid.

As is known, right translations on a Lie group are included in the phase flows of the left-invariant fields. It turns out that the left-invariant fields $u(x,t)$ on the group $\mathrm{SDiff}\, Q$ are the solenoidal fields on Q satisfying the equation

$$\frac{\partial u}{\partial t} = \operatorname{rot}(v \times u). \tag{5.10}$$

The latter expresses the property that the integral curves of the field u are frozen into the flow of the fluid. For an incompressible fluid, the equation (5.10) is equivalent to 'Euler's equation'

$$\frac{\partial u}{\partial t} = [v, u],$$

where $[\,\cdot\,,\cdot\,]$ is the commutator of vector fields. Consequently, again by Noether's theorem, the equations of the geodesics on the group $\mathrm{SDiff}\, Q$ admit an infinite series of linear conservation laws:

$$\langle u, v \rangle = \int_Q (u, v)\, d^3x = \mathrm{const}. \tag{5.11}$$

This statement (established in Appendix 1 of [16]) is a generalization of the old theorem of J.-J. Moreau in which $u = \operatorname{rot} v$.

Since the kinetic energy (5.9) is a non-degenerate quadratic form, the existence of the infinite series of integrals (5.11) gives one the hope of finding the velocity v of the flow as a function on the group SDiff Q. This case differs from the left-invariant case in that the vector fields u in (5.11) are not given a priori but must be found as a solution of the equation (5.10). If this problem is assumed to have been solved, then on the group SDiff Q there naturally arises an infinite-dimensional dynamical system whose phase flow is similar in its properties to a stationary flow of a non-viscous fluid. It would be interesting to study this system from the hydrodynamical point of view described in § 4 (vortex vectors and manifolds, Bernoulli surfaces, invariant measures...). Such an approach might be called *secondary hydrodynamics*. The first (but far from definitive) steps in this direction were made in the paper [23].

§ 6. Energy criteria for stability

As previously noted, Euler was the first to investigate the properties of elastic systems deviating slightly from an equilibrium state under small perturbing actions. The contemporary theory of stability for elastic systems is based on an extension of the classical theory of stability to continuous systems and can be regarded as a part of the theory of differential equations in Hilbert space.

To find conditions for stability of elastic systems in a potential force field, one usually uses the *energy approach* based on the Lagrange–Dirichlet theorem: an equilibrium state is stable if the potential energy has a strict local minimum at this point. The potential energy is a functional of the field of displacements of the particles in the elastic system, and depends also on certain parameters (the load, the length of a rod, flexural rigidity, and so on). The region of stability in the space of parameters is determined by the condition that the second variation of the potential energy is a positive-definite quadratic form. On the boundary of the stability region this quadratic form becomes degenerate, and a bifurcation of equilibrium states generally takes place.

Approximation of continuous systems by finite-dimensional systems is widely used in practice. Already Lagrange replaced an elastic medium by a collection of many small particles interacting elastically with each other. Actually, from a physical point of view such a model is even closer to reality. With this remark taken into account, we discuss some new ideas in the investigation of the stability of the trivial solution to a linear system

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \tag{6.1}$$

which admits a quadratic first integral

$$f = \frac{1}{2}(Bx, x). \tag{6.2}$$

The matrices A and B (B is symmetric) will be assumed to be *non-singular*. Almost all of what is said below can be carried over to the infinite-dimensional case, with the necessary precautions.

Since the system (6.1) is linear, Lyapunov stability of the equilibrium state $x = 0$ means that all its solutions are bounded. The quadratic integral (6.2) can be interpreted as the energy integral. Of course, the criterion of stability of the equilibrium point $x = 0$ can be formulated in terms of the spectral properties of the matrix A. However, with the energy approach in mind it is useful to connect the property of stability also with properties of the quadratic integral (6.2) (for example, with its signature).

Theorem 6.1 [24]. *The following statements are true*:

1) *n is even*;
2) *the spectrum of the matrix A is symmetric with respect to the real and imaginary axes*;
3) $\operatorname{div}(Ax) = \operatorname{tr} A = 0$;
4) *if the index of inertia of the quadratic form* (6.2) *is odd, then the equilibrium state $x = 0$ is unstable*;
5) *the system* (6.1) *has $n/2$ independent quadratic integrals*;
6) *the equilibrium state $x = 0$ is stable if and only if the system* (6.1) *admits a positive-definite quadratic integral.*

Linear Hamiltonian systems possess all those properties. It turns out that this observation is no accident: the system (6.1) is indeed Hamiltonian, and f is the Hamiltonian function. However, this system is not, generally speaking, represented in canonical form. The corresponding symplectic structure is given by the bilinear form

$$\omega(x', x'') = (\Omega x', x''), \qquad \Omega = BA^{-1}.$$

Indeed, this bilinear form is obviously non-degenerate. It is necessary to show that Ω is a skew-symmetric matrix. For the proof we use the equality $A^T B = -BA$, which expresses the property that the quadratic form (6.2) is a first integral of the system (6.1). Thus,

$$A^T = -BAB^{-1}, \qquad (A^T)^{-1} = -BA^{-1}B^{-1}.$$

Therefore,

$$\Omega^T = (A^T)^{-1}B = -(BA^{-1}B^{-1})B = -BA^{-1} = -\Omega.$$

After this observation, we can easily express the system (6.1) in the Hamiltonian form

$$\omega(\dot{x}, \,\cdot\,) = (\Omega\dot{x}, \,\cdot\,) = (Bx, \,\cdot\,) = df(\,\cdot\,).$$

The statement 4) in Theorem 6.1 generalizes Thomson's classical theorem on the impossibility of gyroscopic stabilization of an equilibrium with odd Poincaré degree of instability (with an odd index of inertia of the potential energy). The statement can be represented in a somewhat different form if we introduce the *degree of instability* u of the system (6.1) as the number of eigenvalues of A in the right complex half-plane. Let i^- (i^+) be the negative (positive) index of inertia of the quadratic form f. Then

$$u \equiv i^- \pmod 2. \tag{6.3}$$

Since $i^- + i^+ = n$ and n is even, i^- can certainly be replaced by i^+ in this congruence relation.

However, as was shown in [25], the congruence (6.3) is true even in the more general case when

$$\dot{f} = (BAx, x) \leqslant 0. \tag{6.4}$$

Here n can now also be odd. This case corresponds to systems with dissipation of energy. If the dissipation is *total* (the quadratic form (6.4) is negative-definite), then $u = i^-$ (the Ostrowski–Schneider theorem [26]).

The generalized Thomson's theorem (6.3) can also be extended to certain cases where the matrix A is singular. A vector $x \in \mathbb{R}^n$ is said to be *isotropic* if $(Bx, x) = 0$. Of course, if the form f is positive- or negative-definite, then the only isotropic vector is the zero vector.

Theorem 6.2 [27]. *Suppose that n is even, $|B| \neq 0$, i^- (i^+) is odd, and $Ax \neq 0$ for all isotropic vectors $x \neq 0$. Then the matrix A has a pair of non-zero real eigenvalues $\pm\lambda$ with isotropic eigenvectors.*

The question of estimating the *degree of stability s* is more interesting and complex: this is the number of pairs of purely imaginary eigenvalues of A. Again, we assume that A and B are non-singular matrices.

Theorem 6.3 [28]. *The following estimate is true*:

$$|i^+ - i^-| \leqslant 2l,$$

where l is the number of pairs of purely imaginary eigenvalues of A with Jordan blocks of odd order.

Corollary 1.

$$|i^+ - i^-| \leqslant 2s. \tag{6.5}$$

Corollary 2. *Let i^- (or i^+) be equal to* 1. *Then the spectrum of A consists of a pair of real non-zero numbers $\pm\lambda$ and $(n-2)/2$ pairs of purely imaginary eigenvalues with simple elementary divisors.*

The last statement describes the mechanism of stability loss in a typical case when the total energy loses its positive definiteness. This is directly related to problems Euler considered about loss of stability of equilibrium states of elastic rods.

Let $2i^- \leqslant n$. Then since $i^+ + i^- = n$ and $u + s = n/2$, the inequality (6.5) is equivalent to the inequality

$$u \leqslant i^-. \tag{6.6}$$

This was established in this form earlier in the papers [29] and [30] for Hamiltonian systems of a particular type describing the linear dynamics of mechanical systems in a potential field with additional gyroscopic forces. A. A. Shkalikov's paper [31] contains a refinement of (6.6) based on a generalization of a theorem of Pontryagin on self-adjoint operators acting in spaces with an indefinite metric. Our approach, applied to linear Hamiltonian systems of general type, uses the theory of Williamson normal forms.

§ 7. Problem of two centres in spaces of constant curvature

It turns out that Euler's classical result about integrability of the problem of two fixed centres admits an interesting generalization to spaces of constant curvature. But first some comments are needed concerning gravitation in such spaces.

It is well known that in ordinary three-dimensional Euclidean space the gravitational interaction potential has two fundamental properties. On the one hand, it is a harmonic function (it depends only on the distance and satisfies the Laplace equation), and on the other hand, only this potential and the potential of an elastic spring give rise to central force fields in which all bounded orbits of particles are closed (Bertrand's theorem). These properties can be extended naturally to the more general case of three-dimensional spaces of constant curvature (the three-dimensional sphere S^3 and the Lobachevskii space L^3).

To be specific, let us consider the three-dimensional sphere. Suppose that a material particle of unit mass m is moving in a force field with potential V depending only on the distance between the particle and a fixed point $M \in S^3$. This problem is analogous to the classical problem of motion in a central field. Let ϑ be the length of an arc of a great circle connecting m and M, measured in radians. Then V depends only on the angle ϑ. To determine the gravitational potential on the sphere, the Laplace equation should be replaced by the Laplace–Beltrami equation:

$$\Delta V = \sin^{-2}\vartheta \cdot \frac{\partial}{\partial\vartheta}\left(\sin^2\vartheta \cdot \frac{\partial V}{\partial\vartheta}\right) = 0.$$

This equation can be solved at once:

$$V = -\gamma \cot\vartheta + \alpha; \qquad \alpha, \gamma = \text{const}.$$

The additive constant α is not important; we put $\alpha = 0$ below. For definiteness let us consider the case $\gamma > 0$ (so that the point M attracts the point m). The parameter γ plays the role of the gravitational constant multiplied by the mass of the attracting centre M. In addition to the attracting centre M, this force field has a repelling centre M' at the antipodal point (where $\vartheta = \pi$).

In the general case when V is an arbitrary function of the angle ϑ, the trajectories of the point m lie on two-dimensional spheres S^2 containing M and M'.

It is also natural to consider the generalized Bertrand problem: among all potentials $V(\vartheta)$ find those in whose fields almost all orbits of m on S^2 are closed. Its solution (as in the classical case) is given by the two potentials

$$V_1 = -\gamma \cot\vartheta, \quad V_2 = \frac{k}{2}\tan^2\vartheta; \qquad \gamma, k = \text{const}.$$

The first is the gravitational potential, and the second is an analog of the Hooke's law potential (k is the modulus of elasticity).

The motion of a particle in a central gravitational field is subject to the generalized Kepler's laws. In particular, the orbits of the particle are quadrics on S^2, lines of intersection of the sphere with a second-order cone with vertex at the centre of the sphere.

All these results (from various points of view and in various generalities) were obtained by a number of authors, who often did not know about the work of their predecessors. Apparently, the first (and quite detailed) work on this topic was a forgotten paper by the well-known German geometer Wilhelm Killing. These works were compiled and systematized by A. V. Borisov and I. S. Mamaev in the collection [32].

Let us now consider a generalized problem of two fixed centres. This concerns the motion of a point on a two-dimensional sphere in a field with potential V_1+V_2, where

$$V_1=\gamma_1\cot\vartheta_1,\quad V_2=\gamma_2\cot\vartheta_2;\qquad \gamma_1,\gamma_2=\text{const}.$$

Here ϑ_1 and ϑ_2 are arcs of great circles passing through the respective fixed gravitating centres M_1 and M_2 and a moving particle.

The equations of motion in Hamiltonian form can be solved by separation of variables in orthogonal *spheroconical* coordinates on S^2. This is Killing's result. Our observation is that the problem remains completely integrable if an elastic force is added (with generalized Hooke's law potential) with centre half-way between M_1 and M_2 [33]. This result is quite analogous to Lagrange's addition to the classical Euler problem. All of this is also true for the Lobachevskii space, of course.

What was touched on above is only a small part of Leonhard Euler's creative heritage. We did not even mention his outstanding results in analysis, geometry, and number theory! Recognizing this, let us finish with Laplace's well-known appeal: "Read Euler, study Euler. He is our teacher!"

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