ISSN 1560-3547, Regular and Chaotic Dynamics, 2007, Vol. 12, No. 5, pp. 531–565. © Pleiades Publishing, Ltd., 2007.

Asymptotic Stability and Associated Problems of Dynamics of Falling Rigid Body

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Received February 17, 2007; accepted July 2, 2007

Abstract—We consider two problems from the rigid body dynamics and use new methods of stability and asymptotic behavior analysis for their solution. The first problem deals with motion of a rigid body in an unbounded volume of ideal fluid with zero vorticity. The second problem, having similar asymptotic behavior, is concerned with motion of a sleigh on an inclined plane. The equations of motion for the second problem are non-holonomic and exhibit some new features not typical for Hamiltonian systems. A comprehensive survey of references is given and new problems connected with falling motion of heavy bodies in fluid are proposed.

MSC2000 numbers: 37J60

DOI: 10.1134/S1560354707050061

Key words: rigid body, ideal fluid, non-holonomic mechanics

1. INTRODUCTION

One of the first works by A.M. Lyapunov on the stability theory dealt with stability of steady rotational motions of a rigid body in an unbounded volume of ideal fluid [1]. Not long before the general equations of motion for a rigid body in fluid had been derived by Kirhhoff and the first non-trivial case of integrability was discovered by Clebsëh. The problem of motion of rigid bodies in a fluid became an important theme in research activity of A.M. Lyapunov. A good example is his note on a new special case of motion when the equations of motion can be solved analytically [2]. In some sense, this special case is “reciprocal” to the case found by Steklov (a student and collaborator of A.M. Lyapunov) who corrected some inaccuracies in the Clebsch’s work. It is interesting to note that Lyapunov discovered his case of integrability after he had noticed flaws in Steklov’s calculations [3], who was preparing his master’s thesis under supervision of Lyapunov.

Historical comments. Besides the broadly known works by Lyapunov mentioned above there are two unpublished manuscripts devoted to the analysis of motion of a body in fluid [4, 5]; these handwritten manuscripts are stored in the Archive of the Russian Academy of Science in St. Petersburg[1)]. The manuscript *About integration of the differential equations of motion for a rigid body in fluid* (1893–1895) is a voluminous work concerned the study of meromorphic properties on the plane of complex time of the general solution to the Kirchhoff equations (at present this is known as the Painlevé–Kovalevskaya test or as the Kovalevskaya–Lyapunov method). In a well-known earlier paper [6] Lyapunov applied his method to the analysis of motion of a heavy rigid body with a fixed point. In comparison with the seminal work by S.V. Kovalevskaya, the Lyapunov’s method extensively uses a small parameter and the equation in variations. This method can righteously be considered the parent of the modern methods for proving non-integrability developed by Ziglin, and by Morales

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1)On the occasion of the 150th birthday of A.M. Lyapunov, these and some other of his unpublished works on motion of a rigid body are being currently prepared for publication.

and Ramis [7]. Lyapunov did not complete this manuscript perhaps due to severe analytical difficulties: a complete meromorphic analysis of the Kirhhoff equations remains an open problem at present. Some recent discussions of this topic can be found in [8–10]. In [8] the meromorphic analysis is carried out under some general assumptions.

In his second unpublished work *On the motion of a heavy rigid body in the two cases indicated by Clebsch* (1888-1893) Lyapunov makes an attempt to obtain an analytical solution (in terms of quadratures) for the case of integrability due to Clebsch (in this case some restrictions are imposed on the parameters of the system). Seeking coordinates in which the problem is separable, Lyapunov introduced elliptic coordinates and obtained two polynomial equations of fourth degree. Most likely, this stopped Lyapunov's advance. It should be noted that in spite of the fact that the general solution in the case of Clebsch is considered to be found (good examples are the papers by Kobb [11], Kötter [12, 13] and some other modern papers), no general methods for solving such systems have been invented (i.e. there are no explicit analytical expressions for separating coordinates such as, for example, in the case of Kovalevskaya). In these papers, the stumbling block that stood in the Lyapunov's way seems to be overcome with the help of some tricky manipulations, however, the quadratures for the solution that ensue are useless in the analysis of the behavior of this solution and therefore give very little information (if any) about the body's motion. We hope that some more effective and straightforward methods for introducing separable coordinates will be worked out.

Among the unpublished works of A.M. Lyapunov on dynamics of rigid bodies (which are also stored in the St. Petersburg Archive of RAS) of special note is one of his early papers dealing with motion of a rigid body with multiply connected cavities containing potential flows of an ideal liquid. To a known extent the results obtained in this paper forestalled the master's thesis of N.E. Joukovski, but Lyapunov's investigations were more mathematically involved while Joukovski provided a gyrostatic interpretation of the liquid motion [15]. Two other unpublished essays are concerned with an analytical exploration of the Hess case of the Euler–Poisson equations and the motion of a body on a frictionless plane. It seems that the publications by N.A. Nekrasov [16] and G.V. Kolosov [17], which occurred around that time, motivated his decision to abstain from publishing his results. However, these publications are not as comprehensive and accurate as typical Lyapunov's works.

As is known, Lyapunov's most famous and important research on the theory of stability is his doctoral thesis *The General Problem of the Stability of Motion* [18], which gained him worldwide recognition. It is this thesis that turned the theory of stability into a full-fledged branch of mathematical research. The methods for analysis of stability thoroughly considered in this work are very effective and easy to apply to autonomous systems of differential equations. Stability of non-autonomous systems is essentially harder to explore because even the formulation of the problem of stability can already be a very non-trivial independent task, let alone the problem itself.

One such a non-autonomous system is considered in this paper. We study stability of equilibrium configurations of a mechanical system when the traditional potential energy is multiplied by an unboundedly growing function of time. Such a system occurs naturally in the two following problems: 1) analysis of the falling motion of a heavy rigid body in an unbounded volume of ideal fluid (the formulation of this problem roots back to V.A. Steklov, D.N. Goryachev and S.A. Chaplygin) and 2) analysis of the sliding motion of the non-holonomic Chaplygin sleigh down an inclined plane. Problems of this kind share an intriguing common feature: in the absence of friction the state of equilibrium is asymptotically stable with respect to coordinates and unstable with respect to velocities. To study such problems, new methods (mainly based on change of variables and introduction of effective dissipation) are being elaborated. For some investigations on these problems see also [14, 19].

2. EQUATIONS OF MOTION AND SPECIAL CASES

Let us consider the motion of a rigid body in a homogeneous gravity field in an infinite volume of irrotational incompressible fluid resting at infinity. First let us give general equations of motion of a body in a fluid under the action of an external force field:

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{p} \times \frac{\partial H}{\partial \boldsymbol{p}} + \boldsymbol{K}, \qquad \dot{\boldsymbol{p}} = \boldsymbol{p} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{F}, \tag{2.1}$$

where $\boldsymbol{F}$ and $\boldsymbol{K}$ are the total force and moment applied to the body. These equations go back to Kirchhoff. If the external forces have the potential nature, then Eqs. (2.1) supplemented by the equations for directional cosines and for coordinates of a fixed point in the body, can be written as follows:

$$
\begin{gathered}
\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{p} \times \frac{\partial H}{\partial \boldsymbol{p}} + \boldsymbol{\alpha} \times \frac{\partial H}{\partial \boldsymbol{\alpha}} + \boldsymbol{\beta} \times \frac{\partial H}{\partial \boldsymbol{\beta}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \\
\dot{\boldsymbol{p}} = \boldsymbol{p} \times \frac{\partial H}{\partial \boldsymbol{M}} - \frac{\partial H}{\partial x_1}\boldsymbol{\alpha} - \frac{\partial H}{\partial x_2}\boldsymbol{\beta} - \frac{\partial H}{\partial x_3}\boldsymbol{\gamma}, \\
\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \frac{\partial H}{\partial \boldsymbol{M}}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \frac{\partial H}{\partial \boldsymbol{M}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{M}}, \\
\dot{x}_1 = \left(\boldsymbol{\alpha}, \frac{\partial H}{\partial \boldsymbol{p}}\right), \quad \dot{x}_2 = \left(\boldsymbol{\beta}, \frac{\partial H}{\partial \boldsymbol{p}}\right), \quad \dot{x}_3 = \left(\boldsymbol{\gamma}, \frac{\partial H}{\partial \boldsymbol{p}}\right),
\end{gathered}
\tag{2.2}
$$

where vectors $\boldsymbol{p}$, $\boldsymbol{M}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are the projections of linear momentum, angular momentum, and unit vectors along axes in the fixed frame of reference on the axes connected with the body; and x_1, x_2, x_3 are the projections of position vector of the origin of moving coordinate system on the fixed axes. The Hamiltonian of system (1.2) has the form

$$
\begin{gathered}
H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\mathbf{B}\boldsymbol{M}, \boldsymbol{p}) + \frac{1}{2}(\mathbf{C}\boldsymbol{p}, \boldsymbol{p}) + U, \\
U = \mu\big(x_3 + (\boldsymbol{r}, \boldsymbol{\gamma})\big), \quad \mu = \mu_{\mathrm{b}} - \mu_{\mathrm{f}}, \quad \boldsymbol{r} = \frac{\mu_{\mathrm{b}}\boldsymbol{r}_{\mathrm{b}} - \mu_{\mathrm{f}}\boldsymbol{r}_{\mathrm{f}}}{\mu_{\mathrm{b}} - \mu_{\mathrm{f}}},
\end{gathered}
\tag{2.3}
$$

here, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are symmetric matrices determined by geometry of the body and by its inertial properties, μ_{b}, μ_{f} are the weight of the body and the weight of the displaced fluid, and $\boldsymbol{r}_{\mathrm{b}}$, $\boldsymbol{r}_{\mathrm{f}}$ are the position vectors of the center of gravity and the center of pressure in moving axes. The case $\mu_{\mathrm{b}} = \mu_{\mathrm{f}}$ (a suspended body) will be also studied below.

By straightforward calculations we can verify that there are three integrals of motion (one of which explicitly contains the time):

$$
(\boldsymbol{p}, \boldsymbol{\alpha}) = P_1, \quad (\boldsymbol{p}, \boldsymbol{\beta}) = P_2, \quad (\boldsymbol{p}, \boldsymbol{\gamma}) + \mu t = P_3.
$$

This means that linear momentum of the body + fluid system can be represented in the form

$$
\boldsymbol{p} = P_1\boldsymbol{\alpha} + P_2\boldsymbol{\beta} + (P_3 - \mu t)\boldsymbol{\gamma},
\tag{2.4}
$$

i.e., vector $\boldsymbol{P} = (P_1, P_2, P_3)$ is the initial impulse (impact, according to Chaplygin) in the fixed frame of reference.

By the choice of zero time point (for $\mu_{\mathrm{b}} \neq \mu_{\mathrm{f}}$) and by rotation of the fixed axes we can obtain $P_2 = P_3 = 0$. In what follows, we consider this to be fulfilled.

Substituting (2.4) into equations of motion (2.2), we obtain a self-contained system with respect to $\boldsymbol{M}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, which can be written in the Hamiltonian form:

$$
\begin{gathered}
\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial \bar{H}}{\partial \boldsymbol{M}} + \boldsymbol{\alpha} \times \frac{\partial \bar{H}}{\partial \boldsymbol{\alpha}} + \boldsymbol{\beta} \times \frac{\partial \bar{H}}{\partial \boldsymbol{\beta}} + \boldsymbol{\gamma} \times \frac{\partial \bar{H}}{\partial \boldsymbol{\gamma}}, \\
\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \frac{\partial \bar{H}}{\partial \boldsymbol{M}}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \frac{\partial \bar{H}}{\partial \boldsymbol{M}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial \bar{H}}{\partial \boldsymbol{M}}
\end{gathered}
\tag{2.5}
$$

with the Hamiltonian explicitly depending on time:

$$
\bar{H} = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\mathbf{B}\boldsymbol{M}, P_1\boldsymbol{\alpha} - \mu t\boldsymbol{\gamma}) + \frac{1}{2}\big(\mathbf{C}(P_1\boldsymbol{\alpha} - \mu t\boldsymbol{\gamma}), P_1\boldsymbol{\alpha} - \mu t\boldsymbol{\gamma}\big) + \mu(\boldsymbol{r}, \boldsymbol{\gamma}).
\tag{2.6}
$$

Remark. Equations (2.2) in various but equivalent forms can be found in papers by V.A. Steklov [3], D.N. Goryachev [20], and S.A. Chaplygin [21]. Paper [22], we believe, was the first to reduce them to an elegant nonautonomous form, using the representation (2.4) (in the form of Poincaré equations).

Let us point out some special cases when Eqs. (2.5) can be simplified. They are indicated in [22, 23].

2.1. Motion Without an Initial Impulse [22]

Let the initial impulse be equal to zero: $P_1 = 0$. The equations of motion for $\boldsymbol{M}, \boldsymbol{\gamma}$ in a closed form represent a (nonautonomous) Hamiltonian system on $e(3)$ (see below) with the Hamiltonian

$$\bar{H} = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) - \mu t(\mathbf{B}\boldsymbol{M}, \boldsymbol{\gamma}) + \frac{1}{2}\mu^2 t^2(\mathbf{C}\boldsymbol{\gamma}, \boldsymbol{\gamma}) + \mu(\boldsymbol{r}, \boldsymbol{\gamma}).$$

If, in addition, the body has three planes of symmetry intersecting in the center of gravity, then the Hamiltonian can be simplified further: $\mathbf{B} = 0$, $\boldsymbol{r} = 0$.

2.2. Suspended Body [23]

In [23], Chaplygin also indicated the case when the gravitation is balanced by the Archimedean force (an average density of a body is equal to the density of fluid, and, hence, $\mu_{\mathrm{b}} = \mu_{\mathrm{f}}$), however, the center of gravity of the body does not coincide with the center of gravity of the displaced fluid volume. Thus, the body is under the action of a pair of forces, and its total linear momentum in the fixed frame of reference is conserved, i.e.,

$$\boldsymbol{p} = P_1\boldsymbol{\alpha} + P_2\boldsymbol{\beta} + P_3\boldsymbol{\gamma},$$

where $\boldsymbol{P} = (P_1, P_2, P_3) = \mathrm{const}$. As above, by the choice of fixed axes we can obtain the equality $P_2 = 0$. Thus, in this case the evolution of vectors $\boldsymbol{M}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ can be described by an autonomous Hamiltonian system with the Hamiltonian function

$$\bar{H} = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\mathbf{B}\boldsymbol{M}, P_1\boldsymbol{\alpha} + P_3\boldsymbol{\gamma}) + \frac{1}{2}\big(\mathbf{C}(P_1\boldsymbol{\alpha} + P_3\boldsymbol{\gamma}), P_1\boldsymbol{\alpha} + P_3\boldsymbol{\gamma}\big) + \mu_{\mathrm{b}}(\boldsymbol{r}, \boldsymbol{\gamma}),$$

where $\boldsymbol{r}$ is the vector connecting the center of gravity of the body with the center of pressure.

If the initial impulse is directed along the vertical axis, $\boldsymbol{p} = P\boldsymbol{\gamma}$, Єю ¤тюы■Ўш тхъЄюЁют $\boldsymbol{M}$, $\boldsymbol{\gamma}$ ($\boldsymbol{\gamma}$, then the evolution of vectors $\boldsymbol{M}$, $\boldsymbol{\gamma}$ ($\boldsymbol{\gamma}$ is directed along the field of gravity) is described by a system with Poisson's bracket which is determined by the algebra $e(3)$ (i.e., $\{M_i, M_j\} = \varepsilon_{ijk}M_k$, $\{M_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k$, $\{\gamma_i, \gamma_j\} = 0$), and by the Hamiltonian function

$$\bar{H} = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + P(\mathbf{B}\boldsymbol{M}, \boldsymbol{\gamma}) + \frac{1}{2}P^2(\mathbf{C}\boldsymbol{\gamma}, \boldsymbol{\gamma}) + \mu_{\mathrm{b}}(\boldsymbol{r}, \boldsymbol{\gamma}). \qquad (2.7)$$

We have equations of motion in the explicit form

$$\dot{\boldsymbol{M}} = \frac{\partial H}{\partial \boldsymbol{M}} \times \boldsymbol{M} + \frac{\partial H}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{M}}.$$

In [23] Chaplygin indicated the case when Eqs. (2.7) are integrable with an additional integral of fourth degree in components of an angular momentum. The form of the integral is similar to the Kovalevskaya integral.

2.3. Plane-Parallel Motion [3, 20, 21, 24]

The plane-parallel motion of a rigid body is determined by the invariant relations $M_1 = M_2 = 0$, $\alpha_3 = \gamma_3 = 0$. It is easy to show that the dynamical symmetry of the body with respect to the considered (invariant) plane is a necessary condition for the existence of such motions. This leads to the relations

$$b_{11} = b_{22} = b_{33} = b_{12} = 0, \quad c_{13} = c_{23} = 0.$$

In addition, it can be shown that in this case, by choosing the axes connected with the body, one can obtain the equality $\mathbf{B} = 0$ and the diagonal matrix $\mathbf{C}$. Let an angle of rotation of the moving axes relative to the fixed axes be clockwise countered, as it is shown in Fig. 1, then for the unit vectors of fixed axes we have

$$\alpha_1 = \sin\varphi, \quad \alpha_2 = -\cos\varphi, \quad \gamma_1 = \cos\varphi, \quad \gamma_2 = \sin\varphi.$$

For an angle of rotation we obtain the nonautonomous second-order equation

$$a_3\ddot{\varphi} = (c_1 - c_3)\left(\mu^2 t^2 \sin\varphi\cos\varphi + P_1\mu t\cos 2\varphi - P_1^2\sin\varphi\cos\varphi\right) + \mu(x\sin\varphi - y\cos\varphi), \qquad (2.8)$$

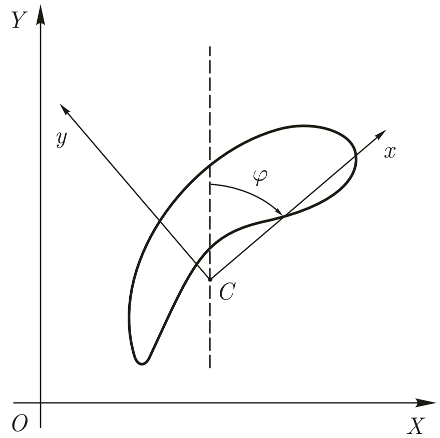


Fig. 1.

where c_1, c_3, a_3 are corresponding elements of diagonal matrices, and $\boldsymbol{r} = (x, y, 0)$.

For a balanced body ($x = y = 0$) without an initial impulse ($P_1 = 0$) we obtain the remarkably simple equation

$$\ddot{\varphi} = kt^2 \sin\varphi \cos\varphi, \quad k = \frac{\mu^2(c_1 - c_3)}{a_3}. \tag{2.9}$$

Remark. In [24–26] this equation is called the Chaplygin equation. In 1890, Chaplygin, being a student, obtained it together with other interesting results. However, he refrained from publishing it. The possible reason was that he could not integrate this equation explicitly. Nevertheless, this work was included in the collected works by Chaplygin, first published in his lifetime (1933, [21]).

Equation (2.9) was also obtained by Goryachev (1893) [20] and Steklov (1894) [3, 27] independently. The latter also observed the simplest properties of solutions of the equation. In particular, Steklov showed that while a body falls down, the amplitude of its oscillations with respect to the horizontal axis decreases, and the oscillation frequency grows. Steklov drew this conclusion in the supplement to his book [3]. In [3], analyzing the asymptotic behavior of a body, he made a series of inaccuracies. The Steklov problem [3, 27] about the asymptotic description of behavior of solutions of the equation was solved by Kozlov [24], who showed that under almost all initial conditions, the motion of a body approaches the uniformly accelerated fall by the wide side upward and it oscillates around the horizontal axis with the increasing frequency of order t and the decreasing amplitude of order $1/\sqrt{t}$. A numerical analysis of asymptotic motions with a different number of half-turns can be found in [28]. Analytic expressions for the asymptotics of a fall were obtained in various forms in [26, 28].

A phenomenon of emerging is described and studied in [29]. Under conditions of a vortex-free flow around a body it is assumed that at the initial moment the wide side of the body is horizontal and the body acquires a horizontal velocity. At subsequent moments the body begins to submerge. However, if its added mass in the transversal direction is sufficiently large, then, further, the body abruptly emerges by the narrow side upward, rising to a greater height than at the initial moment.

2.4. Plane-Parallel Motion of Multiply Connected Body

Let us consider the case of a multiply connected body (the body has reach-through holes). It is well known that the Hamiltonian (2.5) is augmented by the linear terms of the form

$$\Delta H = (\boldsymbol{\sigma}, \boldsymbol{p}) + (\boldsymbol{\zeta}, \boldsymbol{M}). \tag{2.10}$$

Under some restrictions on the body's geometry, its dynamical properties and initial conditions, the body can be set in a plane-parallel motion, similar to that discussed above. Here we confine ourselves to the case discussed in [30] and assume that $\boldsymbol{\zeta} = 0$ and $x = y = 0$ so that rotation of the body is governed by the equation

$$a_3\ddot{\varphi} = (c_1 - c_3)\left(\mu^2 t^2 \sin\varphi\cos\varphi + P_1\mu t\cos 2\varphi - P_1^2\sin\varphi\cos\varphi\right) + \\ \mu t\,(c_1\sigma_1\sin\varphi - c_3\sigma_3\cos\varphi) - P_1(c_1\sigma_1\cos\varphi + c_3\sigma_3\sin\varphi). \quad (2.11)$$

3. THE MOTION OF AN ISOTROPIC BODY

Let us consider the simplest particular case when the equations can be solved by quadratures. The case was pointed out by Steklov [3, 31]. Here

$$\mathbf{A} = \operatorname{diag}(a_1, a_2, a_3), \quad \mathbf{B} = b\mathbf{E}, \quad \mathbf{C} = c\mathbf{E}, \quad \boldsymbol{r} = 0,$$

i.e., the added mass tensor is spherical, however, the body does not have three planes of symmetry, since $\mathbf{B} \neq 0$. (If $\mathbf{B} = 0$, then the motion is trivial: the center of gravity describes a parabola, and the motion of apexes $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ is the same as in the Euler–Poinsot case.)

We isolate equations describing the evolution of angular momentum in the moving frame of reference. They are identical to those in the Euler–Poinsot case:

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \mathbf{A}\boldsymbol{M}.$$

Nevertheless, in order to determine the trajectory of the center of gravity, it is more convenient to rewrite the equations of motion in the fixed frame of reference:

$$\begin{gathered} \dot{N}_1 = b\mu t N_2, \quad \dot{N}_2 = -b\mu t N_1 - bP_1N_3, \quad \dot{N}_3 = bP_1N_2, \\ \dot{x}_1 = bN_1 + cP_1, \quad \dot{x}_2 = bN_2, \quad \dot{x}_3 = bN_3 - c\mu t, \end{gathered} \quad (3.1)$$

where $\boldsymbol{N} = \big((\boldsymbol{\alpha}, \boldsymbol{M}), (\boldsymbol{\beta}, \boldsymbol{M}), (\boldsymbol{\gamma}, \boldsymbol{M})\big)$ is the angular momentum in the fixed frame of reference. It is obvious that the squared angular momentum gives the integral of motion: $\boldsymbol{M}^2 = \boldsymbol{N}^2 = \text{const}$.

If the initial impulse is equal to zero: $P_1 = 0$, then the first three equations in (3.1) are integrable in terms of the elementary functions:

$$N_1 = A\sin(b\mu t^2/2 + \varphi_0), \quad N_2 = A\cos(b\mu t^2/2 + \varphi_0), \quad N_3 = \text{const},$$

where A, φ_0 are arbitrary constants. The body moves along the vertical axis with a constant acceleration: $x_3 = -\dfrac{c\mu t^2}{2}$, and the projection of the trajectory on the plane x_1, x_2 is a spiral which is described by the Fresnel integrals and converges to some fixed point on the plane.

For large times, there holds the asymptotic representation of the form

$$\begin{aligned} x_1 &= x_1^0 - \frac{A}{\mu}\frac{\cos(b\mu t^2/2 + \varphi_0)}{t} + O(t^{-3}), \\ x_2 &= x_2^0 + \frac{A}{\mu}\frac{\sin(b\mu t^2/2 + \varphi_0)}{t} + O(t^{-3}). \end{aligned}$$

If $P_1 \neq 0$, then the equations for $\boldsymbol{N}$ are nonintegrable in terms of the elementary functions. Moreover, on the plane x_1, x_2 a drift appears along the axis Ox_1 with the speed cP_1.

4. QUALITATIVE ANALYSIS OF THE PLANE-PARALLEL MOTION

It was shown above that for a special choice of the moving axes when the kinetic energy is diagonal the angle between the vertical axis and the axis connected with the body (Fig. 1) is described by (2.8), and the motion of the origin of the moving system C (Fig. 1) is described by the equations

$$\begin{aligned} \dot{X} &= (\boldsymbol{\alpha}, \mathbf{C}\boldsymbol{p}) = P_1(c_1\sin^2\varphi + c_2\cos^2\varphi) - \mu t(c_1 - c_2)\sin\varphi\cos\varphi, \\ \dot{Y} &= (\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{p}) = P_1(c_1 - c_2)\sin\varphi\cos\varphi - \mu t(c_1\cos^2\varphi + c_2\sin^2\varphi). \end{aligned} \quad (4.1)$$

Remark. Equation (2.8) corresponds to a *nonautonomous* Hamiltonian system with one degree of freedom. Such systems are studied in more detail in the case when the Hamiltonian is a periodic function of time. In the general case, they demonstrate a chaotic behavior. At the same time, as will be shown below, the dependence of angle φ on time t for system (2.8) is of asymptotic character.

First consider the "simplest" case when a balanced body ($x = y = 0$) falls without an initial impulse ($P_1 = 0$). Then, after the change $2\varphi = \theta$, (2.8) takes the form

$$\ddot{\theta} = kt^2 \sin\theta, \quad k = \frac{\mu^2(c_1 - c_2)}{a_3}. \tag{4.2}$$

In what follows we assume that $c_1 > c_2$, i. e. $k > 0$, and $0 \leqslant \theta < 2\pi$.

4.1. Stationary (Equilibrium) Solutions. Small Oscillations. Doubly Asymptotic Solutions

Equation (4.2) has the simplest "equilibrium" solutions of the form $\theta(t) = \text{const}$:

$$1) \quad \theta = 0, \qquad 2) \quad \theta = \pi. \tag{4.3}$$

The first solution corresponds to the fall by the narrow side downward ($X = X_0,\ Y = Y_0 - \dfrac{\mu c_1 t^2}{2}$), and the second one corresponds to that by the wide side ($X = X_0,\ Y = Y_0 - \dfrac{\mu c_2 t^2}{2}$). Indeed, since $c_1^{-1} < c_2^{-1}$, the angle $\varphi = \pi n$ if the axis Ox is vertical, and $\varphi = \dfrac{\pi}{2} + \pi n$ if the axis Oy is vertical.

Linearizing Eq. (4.2) near the fixed points (4.3), we obtain

$$\begin{aligned} &1) \quad \ddot{\xi} = kt^2\xi, && \theta = \xi, \\ &2) \quad \ddot{\xi} = -kt^2\xi, && \theta = \pi - \xi. \end{aligned}$$

The general solution of these equations is expressed in terms of Bessel function

$$\begin{aligned} &1) \quad \xi(t) = \sqrt{t}\left(C_1 I_{1/4}\left(\sqrt{k}t^2/2\right) + C_2 K_{1/4}\left(\sqrt{k}t^2/2\right)\right), \\ &2) \quad \xi(t) = \sqrt{t}\left(C_1 J_{1/4}\left(\sqrt{k}t^2/2\right) + C_2 Y_{1/4}\left(\sqrt{k}t^2/2\right)\right), \end{aligned} \tag{4.4}$$

where $I_\nu(x)$ and $K_\nu(x)$ are Bessel functions of the second kind, and $J_\nu(x)$, $Y_\nu(x)$ are Bessel functions of the first kind. Thus, in the linear approximation the first solution is unstable, and the second one is (asymptotically) stable with respect to ξ, but not to $\dot{\xi}$. Indeed, using the asymptotics of Bessel functions J_ν and Y_ν for large values of the argument, we find

$$\xi(t) = \frac{A\sin\left(\sqrt{k}t^2/2 + \alpha_0\right)}{\sqrt{t}} + O\big(t^{-5/2}\big), \quad A = \text{const}.$$

Consequently, the amplitude of oscillations decreases similarly to $t^{-1/2}$, and their frequency increases infinitely similarly to t. As mentioned above, this fact was noted in [24].

As shown in [24], using variational methods, one can prove that there exist two solutions $\theta(t)$, $\theta(t_0) = \theta_0$ asymptotic to the unstable position of equilibrium ($\theta = 0$), which approach it from different sides. In addition, because of the invariance of Eq. (4.2) with respect to the change $t \to -t$, there exists a solution $\theta_*(t)$ with the initial data $\theta_*(0) = \pi$ for which [24]

$$\theta_*(t) + \theta_*(-t) = 2\pi, \quad \lim_{t\to-\infty}\theta_*(t) = 0, \quad \lim_{t\to+\infty}\theta_*(t) = 2\pi.$$

Thus, the solution $\theta_*(t)$ is doubly asymptotic (there is also a similar doubly asymptotic solution passing around the circle θ $[0, 2\pi]$ contrariwise). Here the body makes one half-turn. Its trajectory is described by Eqs. (4.1) and shown in Fig. 2a. Note that the upper point of the trajectory is a cusp point: the equation of the curve near this point has the form $Y = \lambda X^{2/3}$, $\lambda = \text{const}$. In Fig. 2b the change of angle φ for this doubly asymptotic solution is shown.

The existence of doubly asymptotic trajectories with an arbitrary number of half-turns was proved in [32].

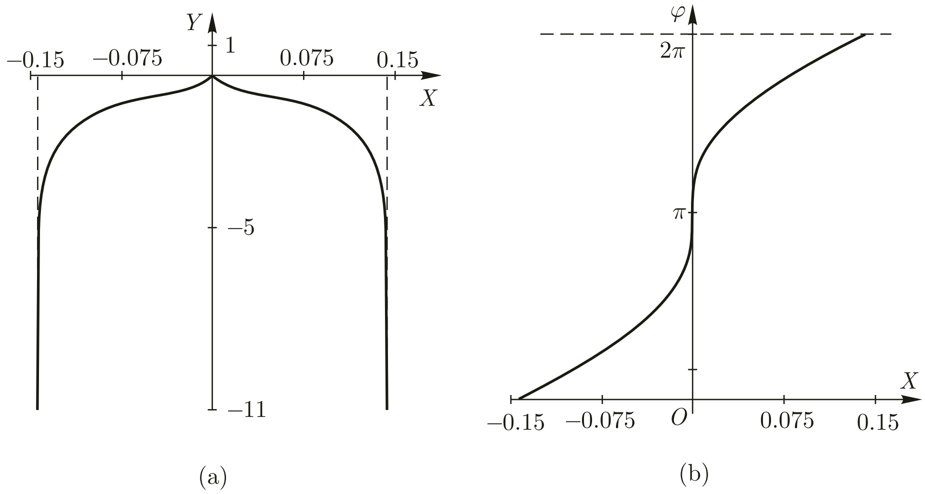


Fig. 2. The trajectory of the body and the value of angle φ depending on the coordinate X for the doubly asymptotic solution with $k = 1$, $\frac{a_3}{\mu} = 0.1$ ($P_1 = 0$); the upper point of the trajectory is singular (see text).

4.2. The Asymptotic Behavior of Solutions of the Chaplygin Equation

As shown in [24] (the idea of the proof in a more general case is presented below), for all solutions of the equation, either $\theta \to 0$ or $\theta \to \pi$ as $t \to \pm\infty$ (i.e., the asymptotic motion of the body is the fall by the wide or narrow side forward).

There is a hypothesis stated in [24], that the measure of trajectories which tend to unstable equilibrium state $\theta = 0(\mathrm{mod} 2\pi)$ as $t \to \pm\infty$, is equal to zero, and thus almost all trajectories tend to $\lim\limits_{t\to\infty} \theta(t) = \pi(\mathrm{mod} 2\pi)$ (i.e., to the fall by the wide side forward).

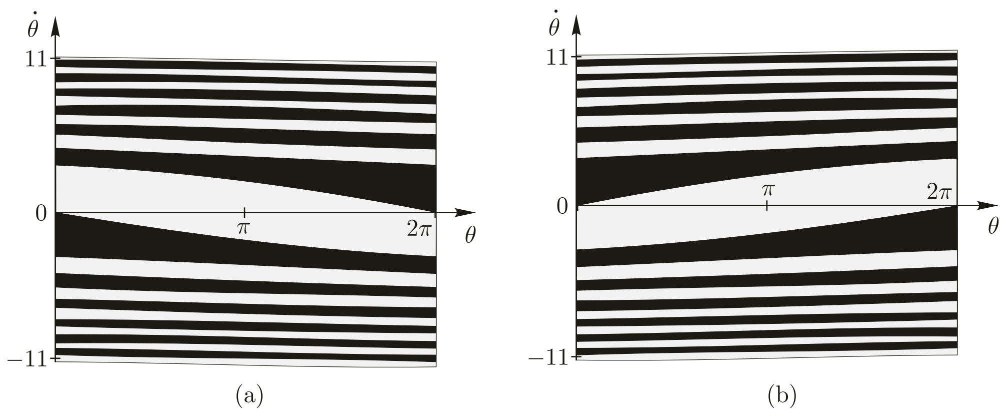


Fig. 3. Domains of the phase plane corresponding to the initial conditions with $t_0 = 0$, when the body makes the same number of half-turns while t changes from 0 to $+\infty$ in the case (a), and while t changes from $-\infty$ to 0 in the case (b) ($k = 1$), the white colour corresponds to an even number of half-turns, the black one corresponds to an odd number.

4.3. Numerical Analysis

On the basis of the statement about the asymptotic behavior, a numerical analysis can be applied to Eq. (4.2) [28]. For this, on the phase plane $(\theta, \dot\theta)$ (more precisely, on the cylinder $\theta \times \dot\theta \in [0, 2\pi) \times (-\infty, +\infty)$) at the initial moment of time $t = t_0$ domains are constructed where the body makes the same number of half-turns as $t \to +\infty$ (or as $t \to -\infty$) before it will be "attracted" to the solution $\theta = \pi$. As seen in Fig. 3a, these domains are located regularly, moreover, their width decreases while $|\dot\theta|$ increases so that for large initial $|\dot\theta|$ only the probability of the fall of the body by the "upper" or "under" side as $t \to +\infty$ makes sense. Boundaries of the domains are filled by initial conditions corresponding to motions asymptotically approaching the unstable positions of equilibrium $\theta = 0,\ 2\pi$. Analogously, one can construct domains corresponding to the same number of half-turns as $t \to -\infty$ (Fig. 3b), moreover, the domains for $t \to +\infty$ turn out to be the mirror images about the line $\theta = \pi$ of the domains for $t \to -\infty$. If we overlay these domains, their boundaries are intersected at points from the line $\theta = \pi$. Doubly asymptotic solutions of Eq. (4.2) with different number of half-turns correspond to them.

Remark. On the cylinder $\theta \times \dot{\theta} \in [0, 2\pi) \times (-\infty, +\infty)$, all boundaries of the domains are glued together in one smooth curve similar to a screw line whose step decreases when $|\dot{\theta}|$ increases. Domains with an even number of half-turns lie on one side of this line, domains with an odd number lie on the other.

Thus, the numerical computations confirm the conjecture that for almost all solutions $\theta(t) \xrightarrow[t\to\pm\infty]{} \pi$, moreover, in the three-dimensional space $t, \theta, \dot{\theta}$, solutions asymptotically approaching unstable equilibrium $\theta = \pi$ fill two-dimensional surfaces. In addition, there is also a countable set of doubly asymptotic solutions that differ by the number of half-turns made when t varies from $-\infty$ to $+\infty$.

In Fig. 4, the trajectories of a body for the doubly asymptotic motions with one and three half-turns are shown.

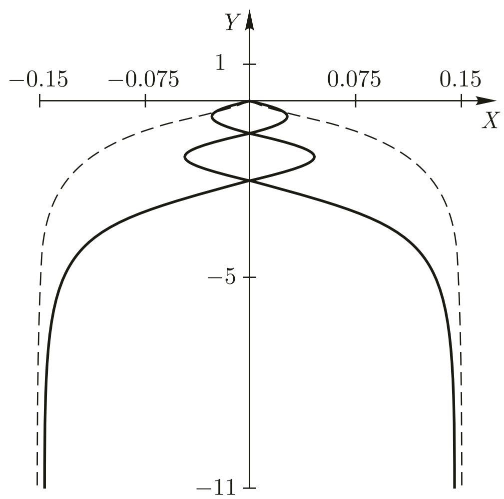


Fig. 4. The trajectory of a body in the case of doubly asymptotic solutions with one (the dotted line) and three (the solid line) half-turns for $k > 0$, $a_3/\mu = 0.1$. There is a singularity in the upper point of the trajectory.

4.4. The Trajectory of the Body

Substituting the asymptotic decomposition for small oscillations (4.4) into Eq. (4.1), after integration we obtain the asymptotic representation for the trajectory of the motion in the form

$$X(t) = A\frac{\cos\left(\sqrt{k}t^2/2 + \theta_0\right)}{\sqrt{t}} + O\left(t^{-3/2}\right), \quad Y(t) = -\mu c_2 t^2 + O\left(t^{-1/2}\right),$$

where A, θ_0 are some constants. Therefore, the trajectory of the motion for large times is close to the sinusoid with the constant step $\Delta y = \dfrac{\pi\mu c_2}{\sqrt{k}}$ and a decreasing amplitude [24]. (Step ΔY is calculated between two consecutive zeros of the function $X(t)$.) The typical trajectory is shown in Fig. 5.

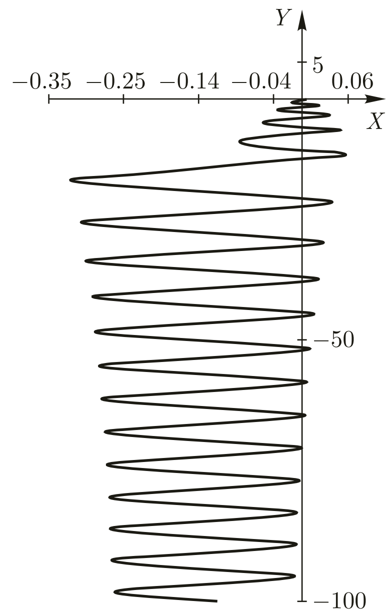


Fig. 5. The typical form of trajectory of a body falling without an initial impulse.

4.5. General Case ($P_1 \neq 0$)

Now let us review the main qualitative features of the behavior of system (2.8), (4.1) in the general case. If $P_1 \neq 0$, there are no longer time-independent solutions similar to (4.3). A statement about the asymptotic behavior also holds in this case. According to this statement, for any solution $\varphi(t)$ of Eq. (2.8) we have

$$1.\ \lim_{t\to+\infty} \varphi(t) = \pi n \quad \text{or} \quad 2.\ \lim_{t\to\infty} \varphi(t) = \frac{\pi}{2} + \pi n, \quad n \in \mathbb{Z}.$$

It also seems that *almost all solutions of the equation tend to a solution of the form* $\varphi(t) = \frac{\pi}{2} + \pi n$ (i.e., the motion of the body approaches the fall by the wide side downward) [24]. Numerical experiments confirm this.

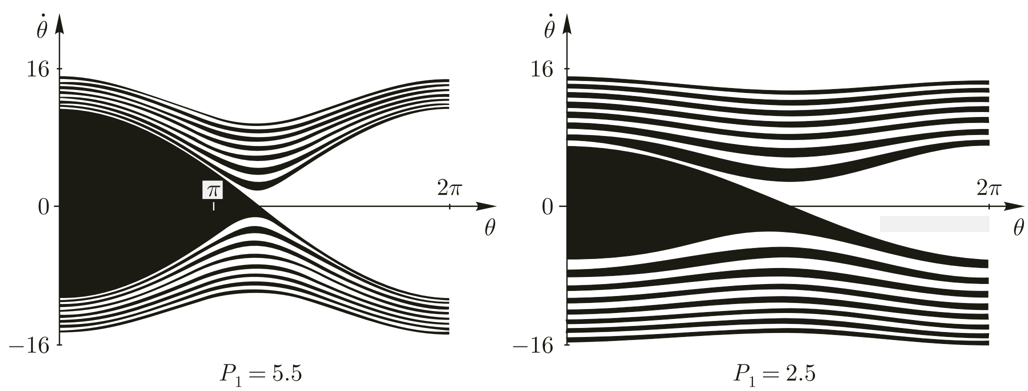


Fig. 6.

Similarly, we can perform a computer analysis, considering those domains on the phase plane at the initial moment $t = t_0$ to which there corresponds the same number of half-turns before the trajectory will be attracted to the solution $\varphi = \frac{\pi}{2}$ as $t \to +\infty$ (Fig. 6). The boundaries of these domains are filled with asymptotic solutions. As in the case $P_1 = 0$, domains corresponding to a different number of half-turns for $t_0 = 0$ and $t \to -\infty$ turn out to be the mirror image of the domains for $t_0 = 0$ and $t \to +\infty$ about the line $\varphi = \dfrac{\pi}{2}$. The intersection points of boundaries of the domains as $t \to \infty$ and $t \to \infty$ correspond to doubly asymptotic solutions.

The typical form of the trajectory of a body thrown at an angle to the horizon is shown in Fig. 7. In Fig. 8 the trajectories are given in the case of doubly asymptotic motions with one and three half-turns of the body.

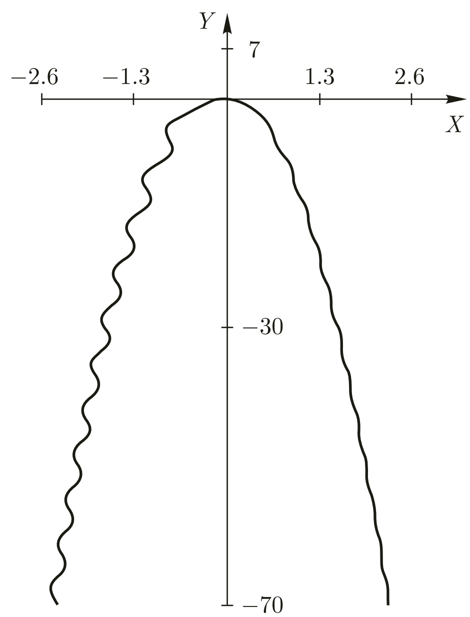


Fig. 7. The typical form of the trajectory of a rigid body thrown at an angle to the horizon.

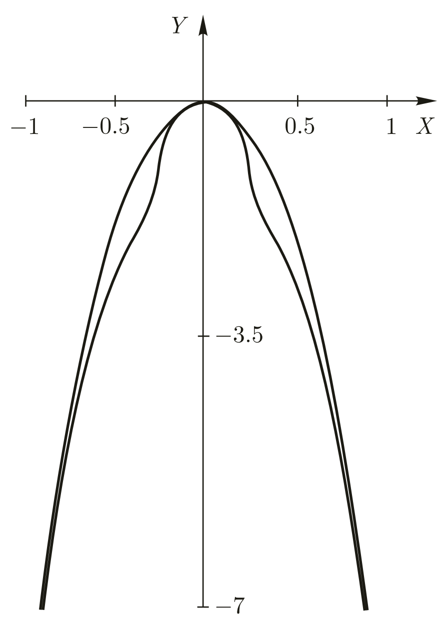


Fig. 8. The trajectories of a body for doubly asymptotic motions with one (the upper curve) and three half-turns.

As shown in [24], in the general case, the asymptotic trajectory of the body is a parabola:

$$X(t) = -P_1 t + o(t), \quad Y(t) = -\frac{\mu t^2}{2c_3} + o(t^2).$$

4.6. Asymptotic Behavior of Multiply Connected Body [30]

Consider the final stage of motion of a multiply connected body with axial symmetry $c_1 = c_3$. Equation (2.11) now reads

$$a_3\ddot{\psi} = k_1 t \sin 2\psi + k_0 \cos 2\psi, \quad \psi = \frac{1}{2}\left(\varphi - \operatorname{arctg}\left(\frac{\sigma_3}{\sigma_1}\right)\right), \tag{4.5}$$

where $k_1 = 2P_1 c_3\sqrt{\sigma_1^2+\sigma_3^2}$, $k_0 = 2P_1 c_1\sqrt{\sigma_1^2+\sigma_3^2}$. The essential difference between this equation and the Chaplygin equation is the absence of terms quadratic in t in the right-hand side.

Nevertheless, it is shown in [30] that practically all methods from the previous analysis can be applied to (4.5). After the change of time $t^{3/2} = \tau$, we get

$$\psi'' + \frac{1}{3\tau}\psi' = \frac{4}{9}k_1 \sin\psi + \frac{4}{9}k_0\frac{1}{\tau^{2/3}}\cos\psi.$$

Arguing identically as in [24], it can be shown that for almost all initial conditions as $t \to +\infty$ the orientation of the body tends to that corresponding to the minimum of the function

$$\overline{\mathcal{E}} = \frac{1}{2}\left(\psi'\right)^2 + \frac{2}{9}k_1\cos^2\psi.$$

Thus, the body's final orientation is governed by the condition that the vector $\boldsymbol{\sigma}$ is directed vertically downwards.

Moreover, the computer analysis from the previous section can naturally be carried over here to show that (as above) the boundaries of the domains with fixed number of half-turns are smooth.

5. A BODY WITH THREE PLANES OF SYMMETRY

As for the case of plane-parallel motion, before studying the general system (2.6), we consider in detail the special case of motion without an initial impulse ($P_1 = 0$) under the additional constraints

$$\mathbf{B} = 0, \quad \boldsymbol{r} = 0. \tag{5.1}$$

Here we obtain a nonautonomous Hamiltonian system (on $e(3)$) for $\boldsymbol{M}$, $\boldsymbol{\gamma}$ with the Hamiltonian

$$\bar{H} = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + \frac{1}{2}\mu^2 t^2(\mathbf{C}\boldsymbol{\gamma}, \boldsymbol{\gamma}). \tag{5.2}$$

In the general case, we can assume that $\mathbf{A}$ is diagonal and $\mathbf{C}$ is arbitrary symmetric.

5.1. Time-Independent (Equilibrium) Solutions and "Normal Oscillations"

The equations of motion for system (5.2) have the form

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \mathbf{A}\boldsymbol{M} + \mu^2 t^2 \boldsymbol{\gamma} \times \mathbf{C}\boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \mathbf{A}\boldsymbol{M} \tag{5.3}$$

and admit the simplest solutions of the form

$$\boldsymbol{M} = 0, \quad \boldsymbol{\gamma} = \pm\boldsymbol{\xi}_i, \quad i = 1, 2, 3, \tag{5.4}$$

where $\boldsymbol{\xi}_i$ are eigenvectors of the matrix $\mathbf{C}$ (for $\mathbf{C}$ degenerate, there are infinitely many eigenvectors $\boldsymbol{\xi}_i$).

Linearizing system (5.3) near solution (5.4), by linear transformations of coordinates one can reduce the equations of motion to the form of "normal oscillations"

$$\ddot{x}_k + t^2\omega_k x_k = 0, \quad k = 1, 2, \tag{5.5}$$

where x_k are corresponding local coordinates near the fixed points $\boldsymbol{\gamma} = \boldsymbol{\xi}_i$. Solutions of system (5.5) can be expressed in Bessel functions (see (4.4)). If all eigenvalues of $\mathbf{C}$ are different, then to the local minimum of the function $V(\boldsymbol{\gamma}) = \frac{1}{2}(\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{\gamma})$ there corresponds an (asymptotically) stable solution of system (5.5) whose asymptotic form for large t is (4.4). The local minimum is defined by some eigenvector of the system. Unstable (now in the linear approximation) solutions correspond to other two eigenvectors.

5.2. The Asymptotic Behavior of Solutions

It turns out that similarly to the plane-parallel case, under arbitrary initial conditions the vector $\boldsymbol{\gamma}$ tends to one of eigenvectors of the matrix $\mathbf{C}$. Indeed, in [22] it is shown that *for any solution $\boldsymbol{\gamma}(t)$ of Eqs.* (5.3)

$$\lim_{t\to\infty} V\big(\boldsymbol{\gamma}(t)\big) = \mathcal{E}_c,$$

where $V(\boldsymbol{\gamma}) = \frac{1}{2}(\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{\gamma})$, and $\mathcal{E}_c$ is a critical value of function $V(\boldsymbol{\gamma})$.

The proof of this statement is based on a representation of Eqs. (5.3) in terms of the new time and new variables. Let us change time and variables by the formulas

$$\frac{1}{2}t^2 = \tau, \quad t\boldsymbol{M} = \boldsymbol{m};$$

here the equations of motion have the form

$$\frac{d\boldsymbol{m}}{d\tau} = -\frac{1}{2\tau}\boldsymbol{m} + \boldsymbol{m}\times\mathbf{A}\boldsymbol{m} + \mu^2\boldsymbol{\gamma}\times\mathbf{C}\boldsymbol{\gamma}, \quad \frac{d\boldsymbol{\gamma}}{d\tau} = \boldsymbol{\gamma}\times\mathbf{A}\boldsymbol{m}. \tag{5.6}$$

It is easy to show that $\operatorname{div}\boldsymbol{v} = -\frac{1}{2\tau}$, i.e., in fact, system (5.6) describes the Kirchhoff equations with dissipation decreasing with time. Consider the energy of "unperturbed" system

$$\mathcal{E} = \frac{1}{2}(\boldsymbol{m}, \mathbf{A}\boldsymbol{m}) + \frac{1}{2}\mu^2(\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{\gamma}). \tag{5.7}$$

Calculating the derivative $\mathcal{E}$ along solutions (5.6), we find

$$\frac{d\mathcal{E}}{d\tau} = -\frac{(\boldsymbol{m}, \mathbf{A}\boldsymbol{m})}{2\tau}.$$

From this equality it easily follows that

1°. $\mathcal{E} \xrightarrow[t\to\infty]{} \mathcal{E}_* = \text{const}$;

2°. the integral $I = \displaystyle\int_{\tau_0}^{\infty} \frac{\big(\boldsymbol{m}(\tau), \mathbf{A}\boldsymbol{m}(\tau)\big)}{2\tau}d\tau$ converges.

The proof is reduced to showing that $\mathcal{E}_* = \mathcal{E}_c$ is the critical value of (5.7), and hence, of $V(\boldsymbol{\gamma})$. It turns out that the assumption $\mathcal{E}_* \neq \mathcal{E}_c$ contradicts the convergence of the integral I.

For the fall of an arbitrary body with three planes of symmetry, there also exists the conjecture [22] that $\mathcal{E}_* = \mathcal{E}_c^{\min}$ *for almost all solutions $\boldsymbol{\gamma}$ of Eqs.* (5.3). Thus, as $t\to\infty$, the body almost always tends to occupy a position in space such that the axis corresponding to the maximal added mass becomes vertical.

5.3. Computer Analysis

The statement about the asymptotic behavior formulated above leads to the natural question about the structure of domains (basins of attraction) corresponding to different asymptotic modes as $t\to\pm\infty$ in the space of initial conditions. Let us choose $t_0 = 0$ and parametrize the joint four-dimensional level of the integrals

$$(\boldsymbol{M}, \boldsymbol{\gamma}) = c = \text{const}, \quad \boldsymbol{\gamma}^2 = 1 \tag{5.8}$$

by Andoyer variables (L, G, l, g) [8, 14, 33] and fix the surface of initial conditions for $t_0 = 0$ by the equations

$$g = g_0, \quad E = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) = \text{const}. \tag{5.9}$$

Depending on the side by which the body falls as $t\to\infty$, we shall paint the point on this surface with a corresponding color. The typical picture is given in Figs. 9 and 10.

We see that the body falls so that the axis corresponding to the maximal added mass is vertical, i.e., it falls either by one wide side downward, or by the other one. This confirms the conjecture formulated above. In addition, generally, the boundary of these domains is fractal, i.e., the surface pattern repeats in smaller parts.

Thus, by analogy with the integrable and nonintegrable (regular and chaotic) systems, the plane-parallel case can be called integrable, and the general case of system (5.2), (5.3) can be called non-integrable. Indeed, in the plane-parallel case the boundaries of domains corresponding to the different orientations of the body are regular, but in system (5.2), (5.3) they are fractal. We shall show below that if system (5.3) has one more additional integral (the Lagrange integral), the boundaries of domains also become regular.

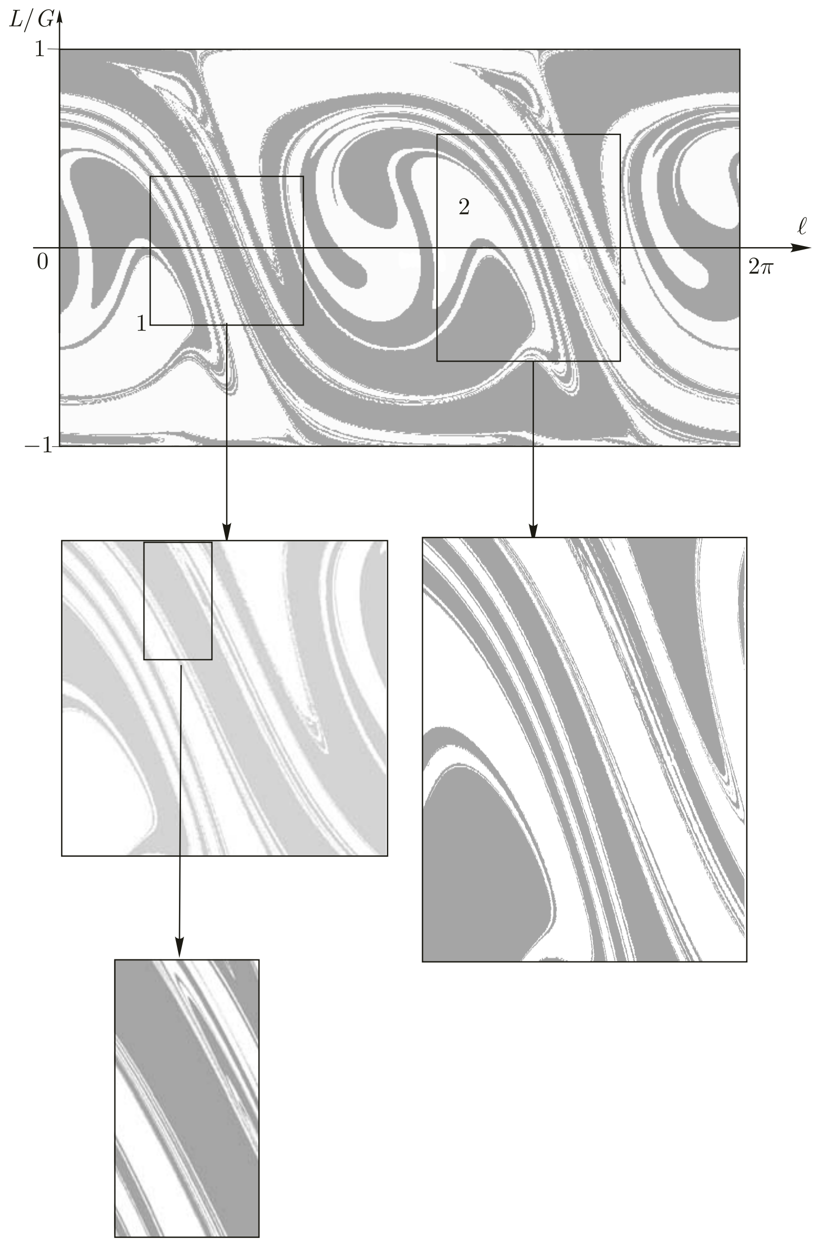


Fig. 9. The typical picture of domains corresponding to two different limit orientations of the body (where the eigenvector corresponding to the greatest added mass is vertical, two colors correspond to its two possible directions) as $t \to +1$. At a fourdimensional level of the first integrals, the given two-dimensional surface is defined by Eqs. (5.8), (5.9). The values of the system parameters are $\mathbf{A} = \mathrm{diag}(1.8; 1.5; 2)$, $\mathbf{C} = \mathrm{diag}(0.5; 2.9; 1.4)$, $\mu = 1$, $(M, \gamma) = 1$, $E_0 = 7$.

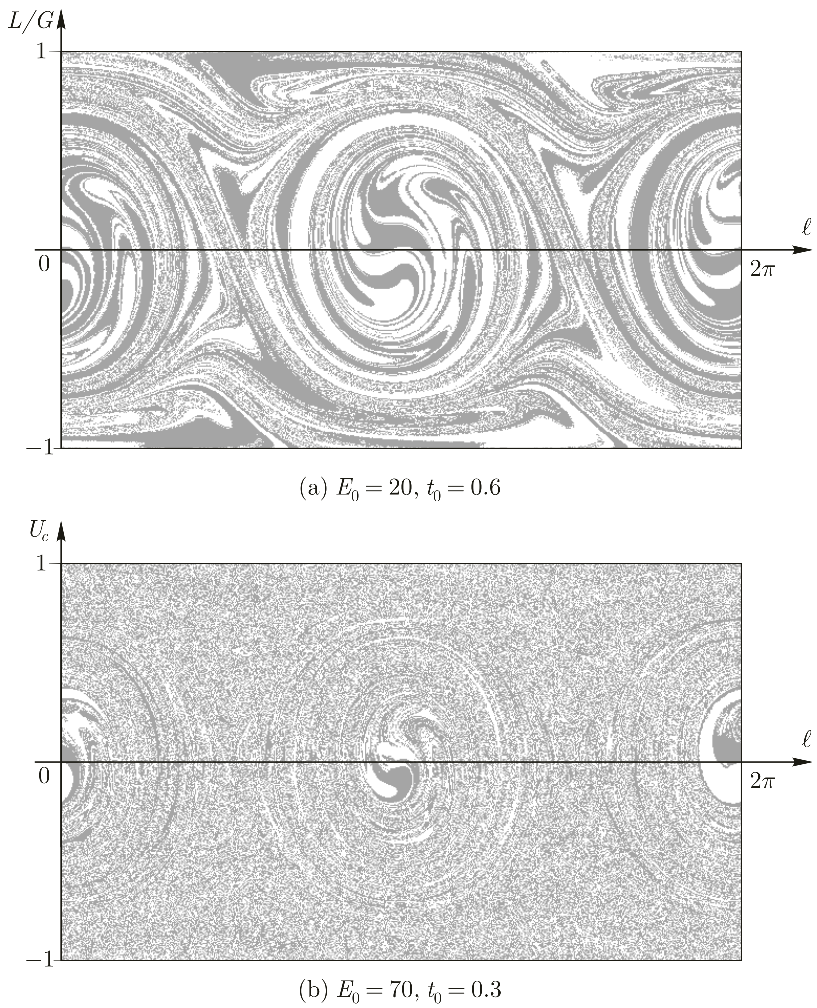


Fig. 10. The typical pattern on the surface of initial conditions according to the behavior of the system as $t \to 1$ when the initial energy and the initial moment t_0 increase. The values of the system parameters are $\mathbf{A} = \mathrm{diag}(1.8; 1.5; 2)$, $\mathbf{C} = \mathrm{diag}(0.5; 2.9; 1.4)$, $\mu = 1$, $(M, \gamma) = 1$.

The fractal structure of boundaries separating the different types of behavior as $t \to \infty$ is closely connected with probabilistic effects arising when descripting asymptotic motions. Indeed, for complex distribution of initial conditions corresponding to different types of asymptotic behavior, under specific (given) initial conditions, the asymptotic behavior becomes unpredictable and only probabilistic description makes sense. This is a kind of probabilistic chaos generated by the structure of initial conditions. The probabilistic description was also proposed by A.I. Neishtadt when he studied the motion around a fixed point of a rigid body under the action of constant and linear (with respect to the angular velocity) dissipative moments [34]. It turned out that for small values of these moments, the dynamics of the system has probabilistic nature. In [34] the explicit formulas for probabilities realizing the evolution of the system to a uniform rotation are obtained. The straight generalization of analytic results [34] to system (5.3), (5.6) is connected with substantial difficulties by virtue of the larger dimension of this system and the dependence of the "dissipation parameter" ε on time: $\varepsilon \sim \dfrac{1}{\tau}$.

Remark. The behavior of a heavy body in a fluid substantially differs from its inertial motion, which is described by Kirchhoff equations. In general case, the last system is nonintegrable [8, 35] and shows the typical chaotic behavior (the Hamiltonian chaos) [33, 36].

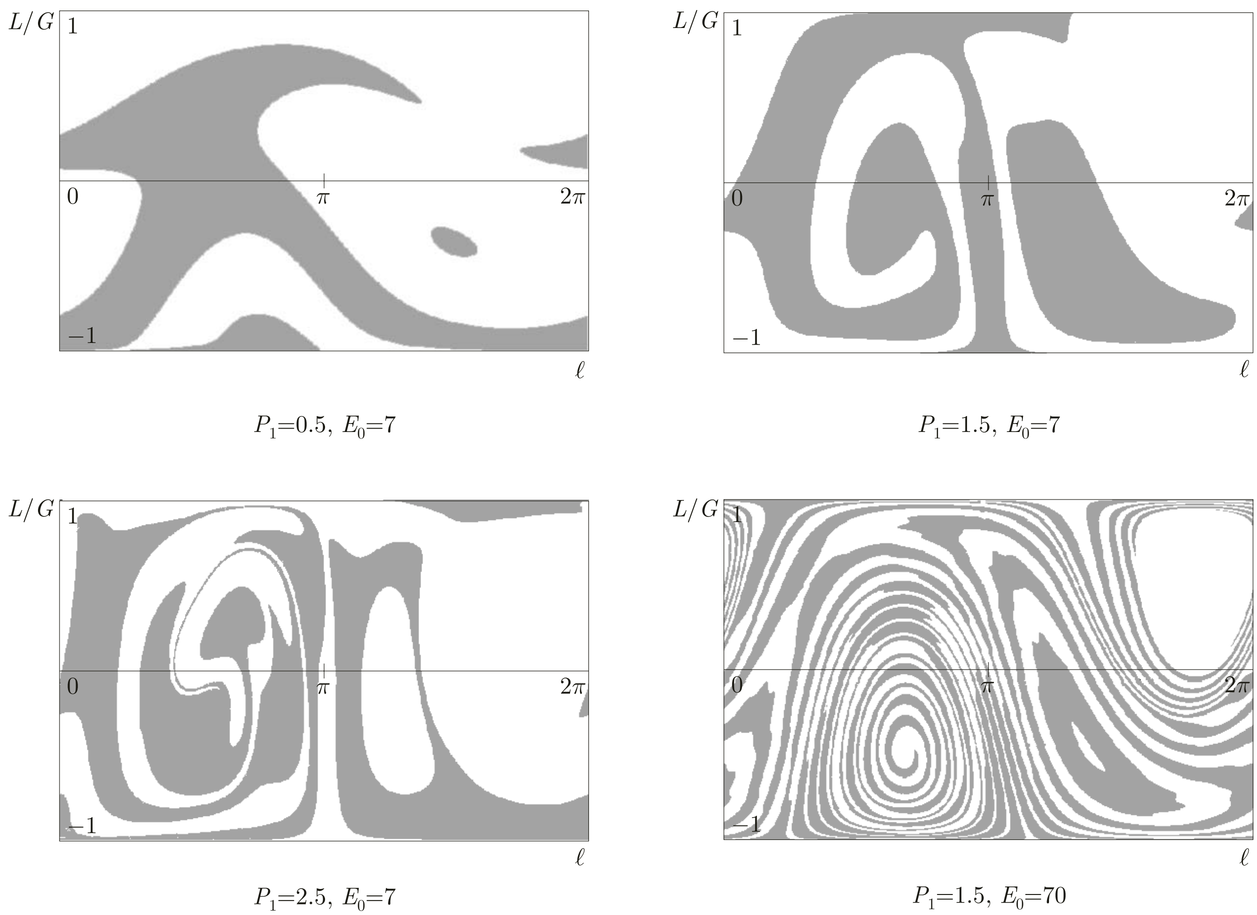


Fig. 11. An analog of the Lagrange case, i.e., the case of existence of the integral $M_3 = \text{const}$. The structure of the basin of attraction is regular ($b_1 = 0.3$, $b_3 = 1.7$, $c_1 = 2.9$, $c_3 = 1.4$, $\mu = 1$).

6. THE FALL OF A BODY WITH SCREW SYMMETRY: STEKLOV SOLUTIONS AND THEIR STABILITY

For the general case $P_1 \neq 0$, $\mathbf{B} \neq 0$ of system (2.6), after the changes

$$\frac{1}{2}t^2 = \tau, \quad \boldsymbol{M} = t\boldsymbol{m}$$

we obtain the equations of motion in the form

$$\begin{gathered}
\frac{d\boldsymbol{m}}{d\tau} = -\frac{1}{2\tau}\boldsymbol{m} + \boldsymbol{m} \times \frac{\partial H}{\partial \boldsymbol{m}} + \boldsymbol{\alpha} \times \frac{\partial H}{\partial \boldsymbol{\alpha}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \\
\frac{d\boldsymbol{\alpha}}{d\tau} = \boldsymbol{\alpha} \times \frac{\partial H}{\partial \boldsymbol{m}}, \quad \frac{d\boldsymbol{\gamma}}{d\tau} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{m}}, \\
H = H_0 + \frac{1}{\sqrt{2\tau}}H_1 + \frac{1}{2\tau}H_2, \qquad (6.1) \\
H_0 = \frac{1}{2}(\boldsymbol{m}, \mathbf{A}\boldsymbol{m}) - \mu(\mathbf{B}\boldsymbol{m}, \boldsymbol{\gamma}) + \frac{\mu^2}{2}(\mathbf{C}\boldsymbol{\gamma}, \boldsymbol{\gamma}), \\
H_1 = P_1(\mathbf{B}\boldsymbol{m}, \boldsymbol{\alpha}) - P_1\mu(\mathbf{C}\boldsymbol{\alpha}, \boldsymbol{\alpha}), \quad H_2 = \frac{P_1}{2}(\boldsymbol{\alpha}, \mathbf{C}\boldsymbol{\alpha}) + \mu(\boldsymbol{r}, \boldsymbol{\gamma}).
\end{gathered}$$

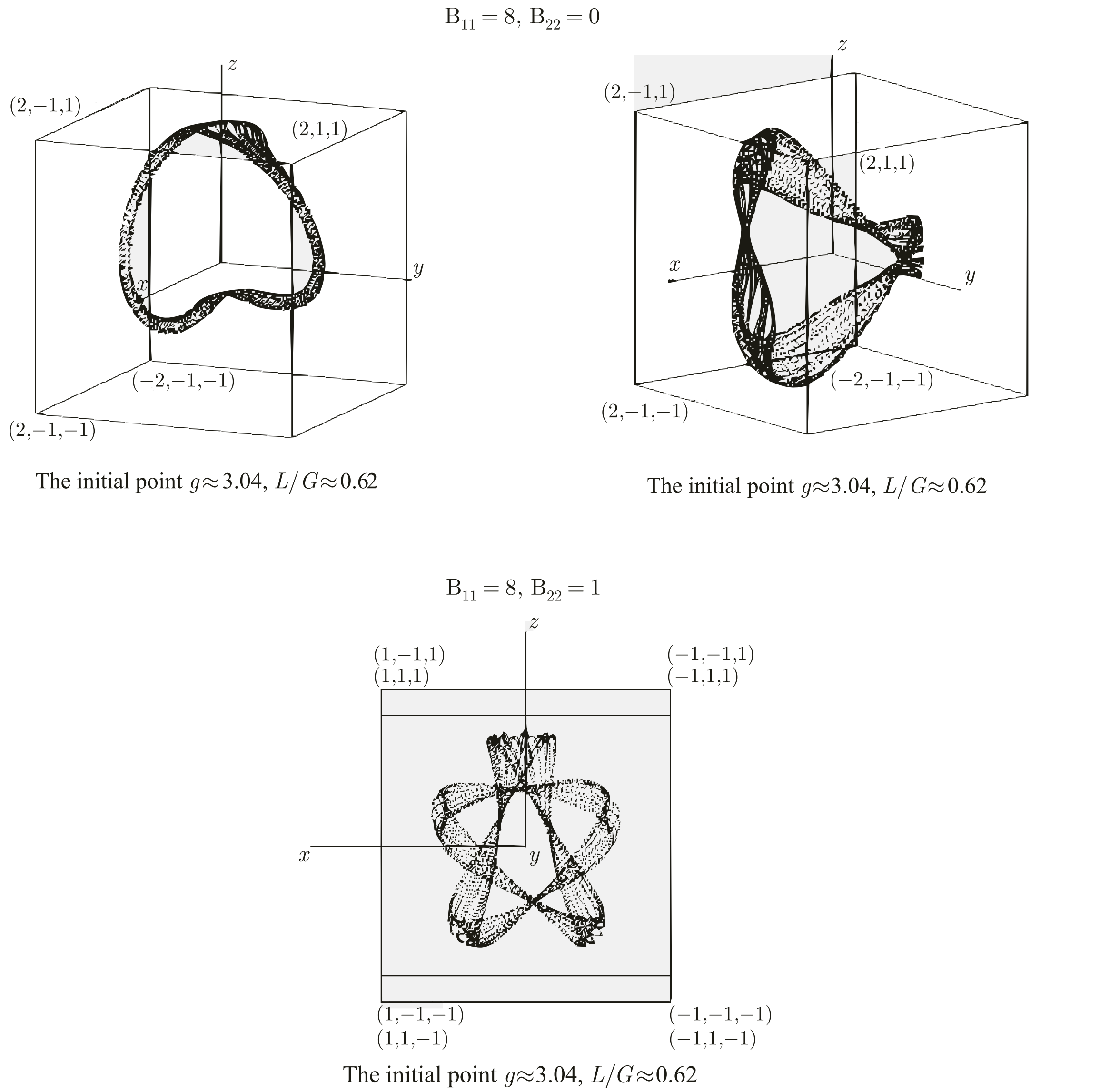


Fig. 12. The typical form of limiting sets in the case of fall of a body with screw symmetry ($A_{11} = 1$, $A_{22} = 1.2$, $A_{33} = 2$, $C_{11} = 1.6$, $C_{22} = 0.1$, $C_{33} = 0$, $P_1 = 0$, $\mu = 1$, $x = y = z = 0$, $E_0 = 3$, $g_0 = \dfrac{\pi}{2}$).

Now, the differentiation of energy along the system gives

$$\frac{dH}{d\tau} = -\frac{(\boldsymbol{m}, \mathbf{A}\boldsymbol{m})}{2\tau} + \frac{1}{2\tau}\left(\mathbf{B}\boldsymbol{m}, \mu\boldsymbol{\gamma} - \frac{2P_1}{\sqrt{2\tau}}\boldsymbol{\alpha}\right) + \frac{W_1}{(2\tau)^{3/2}} + \frac{W_2}{(2\tau)^2},$$
$$W_1 = -P_1\mu(\mathbf{C}\boldsymbol{\alpha}, \boldsymbol{\gamma}), \quad W_2 = P_1^2(\mathbf{C}\boldsymbol{\alpha}, \boldsymbol{\alpha}) + 2\mu(\boldsymbol{r}, \boldsymbol{\gamma}). \tag{6.2}$$

For this system, the asymptotic principles of motion formulated in the previous sections are no longer valid. Moreover, complex attractive regimes of motion different from translational motions exist as $t \to \infty$. First of all, we consider stability conditions (with $\mathbf{B} \neq 0$) for the partial solutions of Eqs. (6.1) corresponding to uniformly accelerated rotations and find domains of values of parameters for which all these solutions lose their stability (and more complex modes become stable). Further, we also consider the case of the zero initial impulse $P_1 = 0$.

6.1. The Linear Stability of Steklov Solutions

For $P_1 = 0$ we separate the equations for $\boldsymbol{m}$, $\boldsymbol{\gamma}$, and the area integral can be represented in the form

$$(\boldsymbol{m}, \boldsymbol{\gamma}) = \frac{\sigma}{\sqrt{2\tau}}, \quad \sigma = \text{const}, \tag{6.3}$$

i.e., $(\boldsymbol{M}, \boldsymbol{\gamma}) = \sigma$.

If, in addition, $\boldsymbol{r} = 0$ and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are simultaneously diagonalizable, then Eqs. (6.1) have partial solutions similar to the time-independent solutions of (5.4). In the basis of eigenvectors of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ we have

$$\gamma_k = \pm 1, \quad \gamma_i = \gamma_j = 0, \quad m_k = \pm\frac{\sigma}{\sqrt{2\tau}}, \quad m_i = m_j = 0, \quad i \neq j \neq k \neq i; \tag{6.4}$$

then there exist six partial solutions in total. Here the body falls so that its axis Oe_k remains vertical, and the angular velocity of rotation around it is determined by the relation

$$\Omega^{(k)} = -\mu b_k t + \sigma a_k,$$

i.e., the rotation of the body is uniformly accelerated. The velocity of the origin of the moving coordinate system in the moving axes is defined by the expression $\boldsymbol{v} = \dfrac{\partial H}{\partial \boldsymbol{p}} = (\sigma\mathbf{B} - \mu t\mathbf{C})\boldsymbol{\gamma}$, whence, using (2.2) we find

$$x_i = \text{const}, \quad x_j = \text{const}, \quad x_k = -\mu c_k \frac{t^2}{2} + \sigma b_k t + \text{const},$$

i.e., the motion of the origin along the vertical axis is uniformly accelerated similarly to the free fall of the body. These uniformly accelerated motions were found out by Steklov [31] (1895) and Chaplygin [21] (1890). In what follows we call them the *Steklov solutions*.

Similarly to solutions (5.4), solutions (6.4) are always unstable in the whole phase space (with respect to variables $\boldsymbol{M}$, $\boldsymbol{\gamma}$). This instability was pointed out by Steklov [31]. At the same time, the stability with respect to positional variables $\boldsymbol{\gamma}$ depends on parameters of the system and requires the special consideration.

To investigate the stability of solutions of (6.4) we choose the new variables

$$v_i = \frac{d\gamma_i}{d\tau}, \quad v_j = \frac{d\gamma_j}{d\tau}, \quad i \neq j \neq k \neq i, \tag{6.5}$$

adding the area integral (6.3) to these equations, we express variables m_i, m_j, m_k in terms of v_i, v_j, σ. Using the relation $\gamma_k = \pm 1 \mp \frac{1}{2}(\gamma_i^2 + \gamma_j^2)$ near solutions of (6.4), we obtain linearized equations for new variables in the form

$$\begin{gathered}
\frac{d\gamma_i}{d\tau} = v_i, \quad \frac{d\gamma_j}{d\tau} = v_j, \\
\begin{aligned}
\frac{dv_i}{d\tau} &= -a_i^{-1} a_j \varkappa_i^{(k)} \gamma_i + a_i^{-1} a_j \mu\big(b_i - b_k + a_i(b_j - b_k)\big) v_j \\
&\quad + \frac{\sigma}{\sqrt{2\tau}} a_i^{-1}\big(\mu a_j a_k (b_i - b_k)\gamma_i + (a_i a_k + a_j a_k - a_i a_j) v_j\big) \\
&\quad - \frac{1}{2\tau}\big(v_i - \sigma^2 a_i^{-1} a_j a_k (a_k - a_i)\gamma_j + \mu(b_j - b_k)\gamma_j\big), \\
\frac{dv_j}{d\tau} &= \ldots,
\end{aligned} \\
\varkappa_i^{(k)} = \mu^2\big(a_i(c_i - c_k) - (b_i - b_k)^2\big),
\end{gathered} \tag{6.6}$$

where the expression for $\dfrac{dv_j}{d\tau}$ is obtained by the change of indices $i \leftrightarrow j$.

We use theorems from [37] about the behavior of the solutions for linear systems of the form $\dfrac{d\boldsymbol{x}}{d\tau} = (\mathbf{A} + \mathbf{V}(\tau))\,\boldsymbol{x}$, where $\displaystyle\int\limits_{\tau_0}^{\infty} |V'(\tau)| d\tau < \infty$ and $V(\tau) \to 0$ as $t \to \infty$. Applying them, we conclude that eigenvalues of linear system (6.6) are expanded in power series of variable $\tau^{-1/2}$

$$\lambda_k(\tau) = \lambda_k^{(0)} + \frac{\lambda_k^{(1)}}{\sqrt{\tau}} + \frac{\lambda_k^{(2)}}{\tau} + \ldots,$$

and the inequalities $\mathrm{Re}\lambda_k^{(0)} \leqslant 0$, where $\lambda_k^{(0)}$ are eigenvalues of system (6.6) for $\tau = \infty$, are the necessary conditions for stability of the system (similarly, $\mathrm{Re}\lambda_k^{(0)} > 0$ are the sufficient conditions for instability). To determine them, we can obtain the (biquadratic) characteristic polynomial

$$\lambda^4 - \lambda^2(\varkappa_i^{(k)} + \varkappa_j^{(k)} - \varkappa_k^{(k)}) + \varkappa_i^{(k)}\varkappa_k^{(k)} = 0. \tag{6.7}$$

Thus, the necessary condition for stability of solutions of (6.4) is that polynomial (6.7) has pure imaginary roots (more precisely, it is the condition of absence of exponential instability with respect to τ). Hence, we find the corresponding constraints on the parameters

$$\begin{gathered} \varkappa_i^{(k)} \cdot \varkappa_j^{(k)} > 0, \quad \varkappa_i^{(k)} + \varkappa_j^{(k)} - \varkappa_k^{(k)} < 0, \\ D = (\varkappa_i^{(k)})^2 + (\varkappa_j^{(k)})^2 + (\varkappa_k^{(k)})^2 - 2\varkappa_i^{(k)}\varkappa_j^{(k)} - 2\varkappa_i^{(k)}\varkappa_k^{(k)} - 2\varkappa_j^{(k)}\varkappa_k^{(k)} > 0. \end{gathered} \tag{6.8}$$

Let us now study in detail the stability of each solution depending on the parameters. Without loss of generality, we put $k = 3$ and $\mu = 1$, $c_3 = 0$, $b_3 = 0$ (the use of integrals $\boldsymbol{\gamma}^2 = 1$ and $(\boldsymbol{M}, \boldsymbol{\gamma}) = \mathrm{const}$ provides the fulfilment of the last two condition). We fix a_1, a_2, a_3, c_1, c_2 and construct on the plane of parameters b_1, b_2 the domains where inequalities (6.8) hold. In this case, relations (6.8) take the form

$$\begin{gathered} (a_1c_1 - b_1^2)(a_2c_2 - b_2^2) > 0, \quad \Phi = a_1c_2 + a_2c_1 + 2b_1b_2 > 0, \\ D = (a_1c_2 - a_2c_1)^2 + 4(a_1b_2 + a_2b_1)(c_2b_1 + c_1b_2) > 0. \end{gathered} \tag{6.9}$$

It is easy to show that there are three qualitatively different cases:

1. $c_3 = 0 > c_1 > c_2$ (i.e. $c_1 < 0$ and $c_2 < 0$); in this case, on the plane b_1, b_2 there are no domains where inequalities (6.8) hold. We can show that there will be either two pairs of real solutions of Eq. (6.7) or four complex solutions.

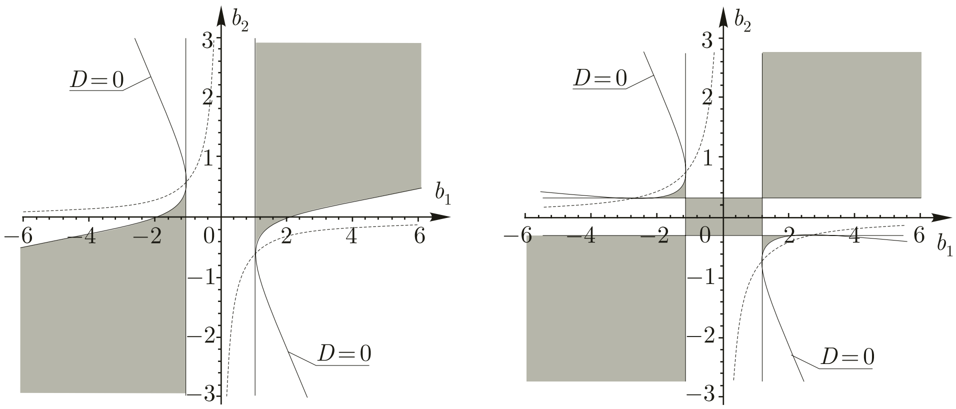
2. $c_1 > c_3 = 0 > c_2$ (i.e. $c_1 > 0$ and $c_2 < 0$); in this case, the domains given by relations (6.9) are located between the lines $b_1 = \pm\sqrt{a_1c_1}$ and the branches of hyperbola given by the relation $D = 0$ (see Fig. 13a).

3. $c_1 > c_2 > c_3 = 0$ (i.e. $c_1 > 0$ and $c_2 > 0$); in this case, the domains given by relations (6.9) are located between the lines $b_1 = \pm\sqrt{a_1c_1}$, $b_2 = \pm\sqrt{a_2c_2}$ and the branches of hyperbola $D = 0$ (see Fig. 13b).

Remark. It can be shown that the curves $\Phi = 0$ and $D = 0$ cross each other at the same points where they cross any line $b_i = \pm\sqrt{a_ic_i}$.

If $b_1 = b_2 = 0$, then conditions (6.9) lead to the conditions mentioned above [22]. Namely, only in case 3, where the axis corresponding to the maximal added mass is vertical, the solution turns out to be stable. Thus, the adding of the matrix $\mathbf{B}$ allows one to stabilize the motion (at least in the linear sense) for which the "middle" axis is vertical, and does not allow one to stabilize the motion for which the "small" axis is vertical.

Now we put $c_1 > c_2 > c_3 = 0$ for definiteness, and plot on the plane of parameters b_1, b_2 the domains of (linear) stability of the Steklov solutions corresponding to the fall by the "wide" and "middle" side; see Fig. 14 (the fall by the "narrow" side is always unstable). It is well seen from the figure that there are domains where all three Steklov solutions are unstable (they are indicated in white).



(a) $c_1 = 1.1$, $c_2 = -0.1$ (b) $c_1 = 1.6$, $c_2 = 0.1$

Fig. 13. The typical pattern of domains on the plane of parameters b_1, b_2 (indicated by the gray color) for which the necessary conditions of stability (6.9) of Steklov solutions hold under different relations between parameters of matrix $\mathbf{C}$. Here, $\mathbf{A} = \mathrm{diag}(1, 1.2, 2)$.

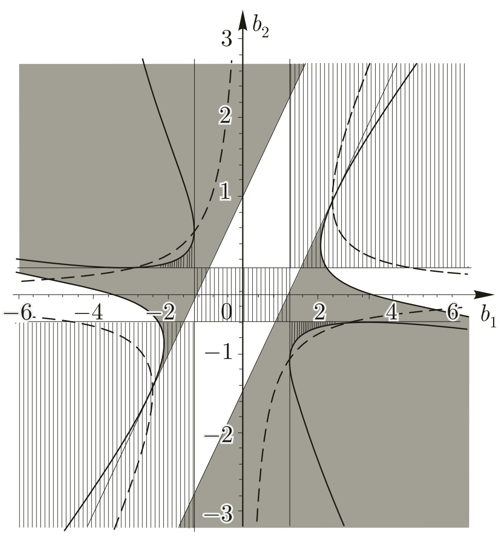


Fig. 14. The typical pattern of domains of stability on the plane of parameters b_1, b_2 of Steklov solutions corresponding to the fall by the wide (i.e., the eigenvector in the direction of the maximal added mass is vertical) and the middle side downward. $\mathbf{A} = \mathrm{diag}(1, 1.2, 2)$, $\mathbf{C} = \mathrm{diag}(1.6, 0.1, 0)$.

6.2. Lyapunov Stability

For one of the Steklov solutions (6.4), namely, for the case when the body falls by the "wide" side downward, one can prove the asymptotic Lyapunov stability.

As mentioned above, without loss of generality, we can put $i = 1$, $j = 2$, $k = 3$ and $b_3 = 0$, $c_1 > c_2 > c_3 = 0$ in (6.4). We construct the Lyapunov function in the form

$$V = H_2 + \frac{1}{\tau} W,$$

where H_2 is the quadratic part of the Hamiltonian near this solution in variables $\gamma_1, \gamma_2, v_1, v_2$:

$$H_2 = \frac{1}{2}(a_2^{-1} v_1^2 + a_1^{-1} v_2^2) + \frac{1}{2a_1}\Big(a_1 c_1 - b_1^2 - \frac{2\sigma}{\sqrt{2\tau}} a_3 b_1 + \frac{\sigma^2}{4\tau^2} a_3 (a_1 - a_3)\Big)\gamma_1^2$$

$$+ \frac{1}{2a_2}\Big(a_2 c_2 - b_2^2 - \frac{2\sigma}{\sqrt{2\tau}} a_3 b_2 + \frac{\sigma^2}{4\tau^2} a_3 (a_2 - a_3)\Big)\gamma_2^2, \tag{6.10}$$

and we look for the function W in the form of a homogeneous quadratic form in γ_1, γ_2, v_1, v_2 with constant coefficients.

It is easy to see that for large τ, the function H_2, and hence, V, is positive definite near the origin under the conditions

$$a_1c_1 - b_1^2 > 0, \quad a_2c_2 - b_2^2 > 0. \tag{6.11}$$

As shown above, these inequalities give us one of the domains of stability for the solution under consideration in the linear approximation (see Fig. 13b). Thus, we can show the asymptotic stability in the domain bounded by inequalities (6.11) for those values of parameters for which we will be able to choose a function V whose derivative along the solutions of a linear system is strictly negative (for sufficiently large τ).

The derivative of function V along solutions of system (6.6) has the form

$$\frac{dV}{d\tau} = -\frac{1}{\tau}G_1 + \frac{1}{\tau^{3/2}}G_2 + \frac{1}{\tau^2}G_3,$$

where G_1, G_2, G_3 are homogeneous quadratic forms in variables $\gamma_1, \gamma_2, v_1, v_2$. Thus, for large τ, the sign of the derivative $\dfrac{dV}{d\tau}$ is determined by the quadratic form G_1 which must be positive definite in the case of the asymptotic stability.

By the straightforward calculations it can be shown that it is necessary to choose W in the form

$$W = k_1v_1\gamma_1 + k_2v_2\gamma_2;$$

then G_2 and G_3 are independent of v_1, v_2 and

$$\begin{aligned} G_1 = {} & 2k_1a_1^{-1}a_2(a_1c_1 - b_1^2)v_1^2 + 2k_2a_2^{-1}a_1(a_2c_2 - b_2^2)v_2^2 + a_2^{-1}(1 - 2a_2k_1)\gamma_1^2 + a_1^{-1}(1 - 2a_1k_2)\gamma_2^2 \\ & + \frac{1}{2}a_1^{-1}\big(b_1 - 2k_1(a_1b_2 + a_2b_1)\big)\gamma_1v_2 - \frac{1}{2}a_2^{-1}\big(b_2 - 2k_2(a_1b_2 + a_2b_1)\big)\gamma_2v_1. \end{aligned} \tag{6.12}$$

It is easy to obtain the conditions for positive definiteness of the form G_1:

$$\begin{gathered} 0 < k_1 < \frac{1}{2a_2}, \quad 0 < k_2 < \frac{1}{2a_1}, \\ -4(a_1b_2 + a_2b_1)^2k_1^2 - 16a_1a_2(c_1a_1 - b_1^2)k_1k_2 - b_1^2 + 4(2a_1a_2c_1 - a_2b_1^2 + a_1b_1b_2)k_1 > 0, \\ -4(a_1b_2 + a_2b_1)^2k_2^2 - 16a_1a_2(c_2a_2 - b_2^2)k_1k_2 - b_2^2 + 4(2a_1a_2c_2 - a_1b_2^2 + a_2b_1b_2)k_2 > 0. \end{gathered} \tag{6.13}$$

There are two cases:

1) $b_1 \cdot b_2 > 0$, then, choosing $k_1 = \frac{1}{2}b_1(a_1b_2 + a_2b_1)^{-1}$ and $k_2 = \frac{1}{2}b_2(a_1b_2 + a_2b_1)^{-1}$, we obtain the diagonal quadratic form (6.12) which is obviously positive definite;

2) $b_1 \cdot b_2 < 0$, in this case the sufficient solvability conditions for inequalities (6.13) are determined by solutions of a quartic equation (and have a rather inconvenient form). At the same time, since only one term is positive in the last two relations (6.13), we can obtain the necessary conditions for the solvability of (6.13) in the form

$$\begin{gathered} \Phi_1 = 2a_1a_2c_1 - a_2b_1^2 + a_1b_1b_2 > 0, \quad \Phi_2 = 2a_1a_2c_2 - a_1b_2^2 + a_2b_1b_2 > 0, \\ b_1b_2 > \max(-a_1c_2, -a_2c_1). \end{gathered} \tag{6.14}$$

In Fig. 15 by the gray color the domain is shown where the necessary conditions (6.14) hold. As can be seen from the figure, for $b_1b_2 < 0$ the domain of asymptotic stability does not coincide with the whole domain of fixed sign of quadratic form (6.10).

Remark. An analysis of the (linear and nonlinear) stability for Steklov solutions was carried out in [38, 39]. In particular, conditions (6.13) were obtained in the form of general inequalities for coefficients without taking into consideration those Steklov solutions for which the relation between stability and instability can be different (see above). Here we carried out a geometric analysis of values of possible parameters for which the conditions of stability (6.13) hold, and drew the conclusion about the existence

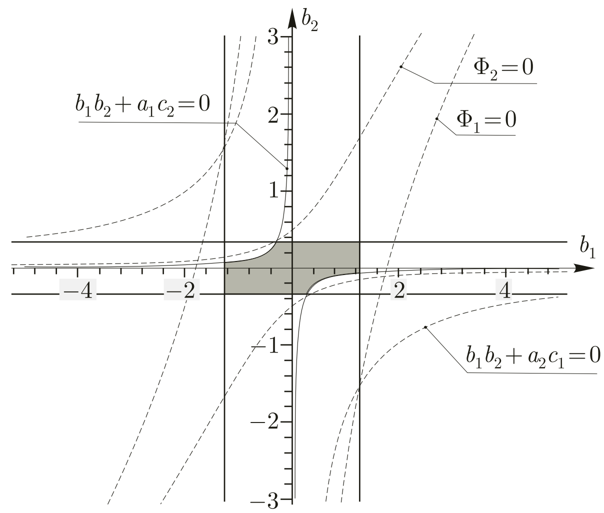


Fig. 15. The domain of asymptotic stability of the Steklov solution corresponding to the fall by the wide side downward with $\mathbf{A} = \mathrm{diag}(1, 1.2, 2)$, $\mathbf{C} = \mathrm{diag}(1.6, 0.1, 0)$.

of a domain of values for the parameters for which all Steklov solutions are unstable. In this case, in the phase space there exists a more complex invariant attracting set of two-dimensional torus type (see Fig. 12) to which the trajectories of system (6.1) tend as $t \to +\infty$. Analytically, the existence of this invariant set remains unproven, because for the present bifurcation theory and qualitative methods have not been developed for systems of type (6.1) for which the linear "dissipation" decreases with time with respect to values of the parameter $\varepsilon \sim \dfrac{1}{\tau}$. In our analysis, simpler conditions for linear stability and Lyapunov stability are also obtained due to the systematic use of the Hamiltonian form of equations of motion.

7. THE CHAPLYGIN SLEIGH ON AN INCLINED PLANE UNDER THE ACTION OF GRAVITY

Consider the non-holonomic problem of motion of a heavy Chaplygin sleigh on an inclined plane. In the context of this problem we study a plane-parallel motion of a body (a plate) under some non-integrable constraint. The non-holonomic system is called *the Chaplygin sleigh*. Suppose that the plate's velocity in the direction perpendicular to its plane is zero. A correct realization of this constraint (first introduced by S.A. Chaplygin [40]) is as follows. A rigid body with attached blade is moving on a perfectly rough plane. It is this blade that prohibits lateral displacement of the body. The blade touches the plane at exactly one point so to make this model more realistic the body is assumed to have two tiny legs that slip without friction. If the center of mass coincides with the point of contact between the blade and the plane then in the literature this system is referred to as a skate or knife edge. Independently this system was studied by Caratheodory [41] in connection with the problem of implementation of constraints by means of viscous friction.

We should note here that one can also use the non-holonomic model of motion of a rigid body to describe the motion of a rigid body in fluid (for example, falling motion in a medium (without parachute effect) [42], falling motion of a plate [43]). In [44] the non-holonomic model of a body's fall is mentioned in connection with the problem of motion in a fluid of missiles with fins. The center of mass of a missile was assumed to move in a predetermined manner thus allowing control over the behavior of the body. However, these works are based on model that provides not exactly correct representation of the non-holonomic constraint within the realm of ideal fluid. A comprehensive discussion of this point can be found in [45–50].

Indeed, suppose that in a certain direction the added mass of a body becomes infinitely large while the other parameters are fixed; then the limit motion of the body is as follows: there is a point in the body whose velocity in this direction is zero. Thus, we have a non-integrable constraint. This, however, does not mean that the motion of the body is governed by the laws of non-holonomic dynamics. The equations of motion are intimately connected with variational principles and the parent model of the motion is *the vaconomic dynamics*. From this point of view, the problem of sliding of a heavy Chaplygin sleigh down an inclined plane can be treated as a *non-holonomic model* of the falling motion of a body in a fluid. It should be noted that this model (contrary to what is stated in [42]) is not connected with the parachute effect and does not realistically describe the interaction between the rigid body and the ambient fluid (see also [43, 44]).

Conversely, vaconomic dynamics can not be systematically used (e.g. as in [51, 52]) to describe the non-holonomic problem that results from a different mechanism of realization of constraints, when instead of an added mass it is the sliding friction coefficient that goes to infinity (it is assumed that prior to the passage to the limit a “liberated” system with viscous friction was considered). Thus the problem of motion of the Chaplygin sleigh needs an absolutely special mechanical interpretation, not connected with the falling motion of a body in fluid. We do not discuss here possible interpretations, which (we believe) must incorporate fluid’s viscosity and circulatory motion around the body. This discussion is a matter of independent research that should focus on a more thorough analysis of the passage to the limit from [53]. However, being essentially different, these two problems are common in respect of the analysis of stability of uniformly accelerated motions and for this reason are considered here together.

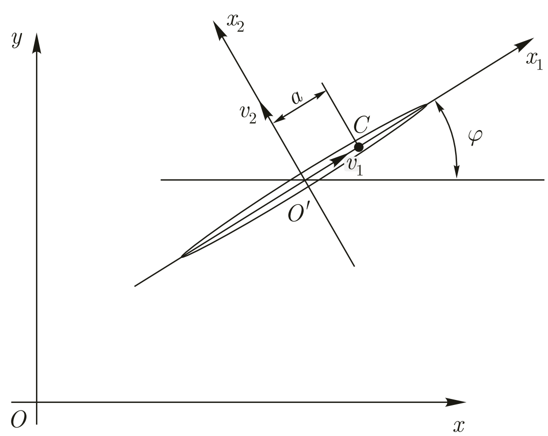


Fig. 16.

So, let the plate’s velocity in the direction normal to its plane be zero. We introduce two coordinate systems: one of them, Oxy, is fixed in space and the other $O'x_1x_2$ is fixed to the plate and has origin at its center (Fig. 16); a is the distance from the point O' to the center of mass (which can be different from zero for plates composed of non-homogenous material).

Let ω be the angular velocity of the body and $(v_1, v_2) = \boldsymbol{v}$ represent the projections of the velocity of O' on the axes of the fixed coordinate system. Then the constraint mentioned above (the projection of the velocity of O' on the axis $O'\eta$ is zero) reads

$$v_2 = 0. \tag{7.15}$$

This constraint is not integrable, hence no restrictions are imposed on the plate’s location in the plane. Using the d’Alembert–Lagrange principle for equations with constraints and eliminating the undetermined multiplier, we get the equations of motion of the form

$$\begin{gathered} m\dot{v}_1 = ma\omega^2 - \frac{\partial U}{\partial x}\cos\varphi, \qquad (I + ma^2)\dot{\omega} = -ma\omega v_1 - \frac{\partial U}{\partial \varphi} \\ \dot{\varphi} = \omega, \qquad \dot{x} = v_1\cos\varphi, \qquad \dot{y} = v_1 \sin\varphi. \end{gathered} \tag{7.16}$$

Here m and I are the mass and the moment of inertia about the center of mass, $\boldsymbol{r} = (x, y)$ is the radius-vector from O to O', φ is the angle of rotation of the axes of the body-fixed frame (Fig. 16), and U is the potential energy of the external force field. It was shown in [54] that equations (7.16) do not have an invariant measure, except for the case $a = 0$. Lack of an invariant measure is a typical feature of non-holonomic systems. Owing to this feature, non-holonomic systems often exhibit non-Hamiltonian asymptotic behavior or even have *strange attractors* in the phase space [55]. In this sense non-holonomic systems more closely resemble dissipative than Hamiltonian systems (a generic non-holonomic system tends to a steady state).

Equations (7.16) preserve the energy

$$\mathcal{E} = \frac{1}{2}(mv_1^2 + (I + ma^2)\omega^2) + U(x, y, \varphi). \tag{7.17}$$

Assuming that the acceleration due to gravity is along the axis Ox, we get

$$U = \mu m(x + a\cos\varphi),$$

where μm is the body's weight. Since the substitution $a \to -a$, $v_1 \to -v_2$, $\varphi \to \varphi + \pi$ does not change the equations of motion, it follows that with no loss of generality we can take $a \geqslant 0$.

We introduce the non-dimensional variables and time

$$v_1 = \sqrt{\mu a A}v, \quad \omega = \sqrt{\frac{\mu}{aA}}w, \quad x = aA\overline{x}, \quad y = aA\overline{y}, \quad \overline{t} = \sqrt{\frac{\mu}{aA}}t, \tag{7.18}$$

where $A^2 = \frac{I+ma^2}{ma^2}$. The equations of motion for the plate become

$$\begin{gathered}\frac{dv}{d\overline{t}} = \varepsilon w^2 - \cos\varphi, \quad \frac{dw}{d\overline{t}} = -\varepsilon vw + \varepsilon\sin\varphi, \quad \frac{d\varphi}{d\overline{t}} = w,\\ \frac{d\overline{x}}{d\overline{t}} = v\cos\varphi, \quad \frac{d\overline{y}}{d\overline{t}} = v\sin\varphi,\end{gathered} \tag{7.19}$$

where $A^{-1} = \varepsilon$, and $0 \leqslant \varepsilon < 1$. Note that for $\mu \neq 0$ the substitution (7.18) is not singular, moreover, if $a \to 0$, then $\varepsilon \to 0$, but $\lim\limits_{a\to 0} aA = \sqrt{I/m}$.

In terms of new variables the energy reads

$$\overline{\mathcal{E}} = \frac{1}{2}(v^2 + w^2) + \overline{x} + \varepsilon\cos\varphi.$$

7.1. Integrable Cases

There are two "extreme" cases $\varepsilon = 0$ and $\varepsilon = 1$ when the system of equations (7.19) is integrable in terms of quadratures.

Let $\varepsilon = 0$ (there is no gravity), then the solution can be expressed in terms of elementary functions as follows:

$$\begin{gathered}w = w_0, \quad \varphi = \varphi_0 + w_0\overline{t}, \quad v = v_0 - w_0^{-1}\sin\varphi,\\ \overline{x} = \overline{x}_0 + w_0^{-1}\Big(v_0\sin\varphi - \frac{1}{2}w_0^{-1}\sin^2\varphi\Big),\\ \overline{y} = \overline{y}_0 - \frac{1}{2}w_0^{-1}\overline{t} - w_0^{-1}\Big(v_0\cos\varphi - \frac{1}{2}w_0^{-1}\cos\varphi\sin\varphi\Big),\end{gathered} \tag{7.20}$$

where v_0, w_0, φ_0, $\overline{x}_0$, $\overline{y}_0$ are constants due to integration. To solve the equations in this case Chaplygin used the reducing multiplier method [40]. However, the quasi-coordinates he introduced (arc length of the trajectory traced by the blade) seriously complicated the derivation of the quadratures, which could have been obtained more trivially. If the body is released with zero initial velocity ($v_0 = 0$), then it moves in a cycloid [42].

Thus, for $\varepsilon = 0$ the body does not fall unboundedly downwards but drifts horizontally with average velocity $\frac{1}{2}\sqrt{\frac{\mu}{aA}}w_0^{-1}$.

Let $\varepsilon = 1$, then the variables

$$\alpha = w\sin\varphi - v\cos\varphi, \quad \beta = w\cos\varphi + v\sin\varphi$$

satisfy the equations $\frac{d\alpha}{d\overline{t}} = 1$ and $\frac{d\beta}{d\overline{t}} = 0$, therefore

$$\alpha = \alpha_0 + \overline{t}, \quad \beta = \beta_0,$$

where α_0, β_0 are constants due to integration. With no loss of generality put $\alpha_0 = 0$, then $\xi = \operatorname{tg}\frac{\varphi}{2}$ satisfies the following non-autonomous equation

$$\frac{d\xi}{d\overline{t}} = \frac{1}{2}\beta_0(1-\xi^2) + \overline{t}\xi. \tag{7.21}$$

The solution to this equation can be expressed in terms of Whittakker functions [56]. Since this solution is very bulk, we do not report it here. Instead, let us consider its asymptotic for $\overline{t} \to \pm\infty$. From equation (7.21) we get $\xi(\overline{t}) = \frac{2}{\beta_0}\overline{t} + O(\overline{t}^{-1})$ and therefore

$$\varphi(\overline{t}) = \pi - \frac{\beta_0}{\overline{t}} + O(\overline{t}^{-3}), \quad v(\overline{t}) = \overline{t} + O(\overline{t}^{-1}),$$
$$\overline{x}(\overline{t}) = -\frac{1}{2}\overline{t}^2 + O(\overline{t}), \quad \overline{y}(\overline{t}) = \beta_0\overline{t} + O(1).$$

This means that asymptotically the body moves in a parabola.

Thus for $\varepsilon = 1$, every motion of the body is asymptotically a uniformly accelerated fall.

7.2. Partial Solutions and Their Stability

The system of equations (7.19) admits two trivial solutions for which $\overline{y} = \text{const}$ and $w = 0$:

$$\varphi = 0, \quad v = v_0 - \overline{t}, \quad \overline{x} = v_0\overline{t} - \frac{1}{2}\overline{t}^2, \tag{7.22.1}$$

$$\varphi = \pi, \quad v = v_0 + \overline{t}, \quad \overline{x} = -v_0\overline{t} - \frac{1}{2}\overline{t}^2. \tag{7.22.2}$$

These solutions represent a rectilinear uniformly accelerated fall of the plate along the vertical axis with the plane of the plate remaining parallel to this axis; for ($\varphi = 0$) the center of mass is above the plate's midpoint and for ($\varphi = \pi$) — below.

To study the stability of these partial solutions (with respect to part of the variables) we introduce new velocities and time in the first three equations (7.19)

$$\overline{t}\,d\overline{t} = d\tau, \quad v = \overline{t}\overline{u}, \quad \omega = \overline{t}\overline{\omega}.$$

A system of non-autonomous equations results

$$\frac{d\overline{u}}{d\tau} = \varepsilon\overline{\omega}^2 - \frac{1}{2\tau}(\overline{u} + \cos\varphi), \quad \frac{d\overline{\omega}}{d\tau} = -\varepsilon\overline{u}\,\overline{\omega} - \frac{1}{2\tau}(\overline{\omega} - \varepsilon\sin\varphi), \quad \frac{d\varphi}{d\tau} = \overline{\omega}. \tag{7.23}$$

The solution (7.22.1) corresponds to a partial solution of (7.23) of the form $\varphi = 0$, $\overline{\omega} = 0$, $\overline{u} = -1 + \frac{v_0}{\sqrt{2\tau}}$; in the phase space this is a straight line tending to the fixed point $\overline{u} = -1, \overline{\omega} = 0$, and $\varphi = 0$.

Similarly, the solution (7.22.2) corresponds to a partial solution of (7.23) of the form $\varphi = \pi$, $\overline{\omega} = 0$, $\overline{u} = 1 + \frac{v_0}{\sqrt{2\tau}}$, which is a straight line tending to the fixed point $\overline{u} = 1, \overline{\omega} = 0$, and $\varphi = \pi$.

The following theorem on nonlinear asymptotic stability holds true.

Theorem 1. *The fixed point $\overline{u} = -1$, $\overline{\omega} = 0$, $\varphi = 0$ (fall with the center of mass above the midpoint) is unstable; the fixed point $\overline{u} = 1$, $\overline{\omega} = 0$, $\varphi = \pi$ (the center of mass below the midpoint) is asymptotically stable.*

Proof. Using polar coordinates

$$\overline{u} = \cos\psi, \quad \overline{\omega} = \rho\sin\psi,$$

we can rewrite equations (7.23) as follows:

$$\begin{aligned} \frac{d\rho}{d\tau} &= -\frac{1}{2\tau}(\rho + \cos\varphi\cos\psi - \varepsilon\sin\varphi\sin\psi), \\ \frac{d\psi}{d\tau} &= -\varepsilon\rho\sin\psi + \frac{1}{2\tau}\frac{\cos\varphi\sin\psi + \varepsilon\sin\varphi\cos\psi}{\rho}, \\ \frac{d\varphi}{d\tau} &= \rho\sin\psi. \end{aligned} \tag{7.24}$$

In polar coordinates the first fixed point transforms into $\rho = 1$, $\psi = \pi$, $\varphi = 0$; to prove the instability of this point, put $\rho = 1 + X$, $\psi + \varepsilon\varphi = \pi + Y$, $\varphi = Z$. Consider then a Chetaev function [57] of the form

$$\Phi = (\psi - \pi)(\psi + 2\varepsilon\varphi - \pi) = (Y + \varepsilon Z)(Y - \varepsilon Z). \tag{7.25}$$

In the rectangle $-\frac{\varepsilon\pi}{2} < Y < \frac{\varepsilon\pi}{2}$, $-\frac{\pi}{2} < Z < \frac{\pi}{2}$ the function Φ is uniquely defined (however Φ is not defined globally on the whole phase space because it is not 2π-periodic in φ and ψ).

Differentiating (7.25) gives

$$\frac{d\Phi}{d\tau} = 2\varepsilon^2(1+X)G_0 + \frac{1}{\tau}(1+X)^{-1}G_1,$$
$$G_0 = Z\sin(Y - \varepsilon Z), \quad G_1 = -Y\big(\sin(Y-\varepsilon Z)\cos Z + \varepsilon\cos(Y - \varepsilon Z)\sin Z\big).$$

For any point inside the rectangle such that $\Phi < 0$ we have $G_0 < 0$; however G_1 is not negative everywhere. In the vicinity of the origin the function $G_{0,1}G_{0,1}$ allows the following representation

$$G_0 = Z(Y - \varepsilon Z) + O(R^3), \quad G_1 = -Y^2 + O(R^3), \quad R = \sqrt{(X^2 + Y^2 + Z^2)}.$$

Therefore, there exists a neighborhood of the origin where $\Phi < 0$, and for sufficiently large τ the derivative $\frac{d\Phi}{d\tau} < 0$. Thus, according to the Chetaev theorem [57] the fixed point is unstable.

To prove the stability of the second fixed point, put $\rho = 1 + X$, $\psi + \varepsilon\varphi = \varepsilon\pi + Y$, $\varphi = \pi + Z$. In a neighborhood of the fixed point, we take the following positive-definite quadratic form as a Lyapunov function:

$$V = \frac{1}{2}\Big(X^2 + Y^2 + (Y + \varepsilon Z)^2 + \frac{\varepsilon}{2\tau}Z^2\Big). \tag{7.26}$$

Lacking 2π-periodicity in φ and ψ, this function is well defined only in the vicinity of the point under consideration.

Differentiating (7.26) gives

$$\begin{gathered} \frac{dV}{d\tau} = -\varepsilon(1+X)F_0 - \frac{1}{\tau}F_1 - \frac{1}{\tau^2}\frac{\varepsilon}{4}F_2, \\ F_0 = Y\sin Y, \quad F_2 = Z^2, \quad F_1 = \frac{1}{2}\big(X^2 + Y^2 + (Y + \varepsilon Z)^2\big) + O(R^3), \end{gathered} \tag{7.27}$$

hence in some neighborhood of the fixed point $\frac{dV}{d\tau} < 0$, which proves, according to the Lyapunov theorem, the stability of the fixed point.

Since the function G_0 is degenerate, proof of asymptotic stability implies more delicate analysis. We will use the method from [24]. It follows from (7.26) and (7.27) that there is a neighborhood of the fixed point where V and F_1 have no critical points except the origin of coordinates and $G_1 > 0$ everywhere but $\boldsymbol{R} = 0$. Hence for all initial conditions (for $\tau = \tau_0$) from this neighborhood the function $V(\tau)$ decreases monotonically and by virtue of the fact that V is bounded from below we conclude that $V(\tau) \xrightarrow[\tau\to\infty]{} V^*$. Assume that $V^* > 0$. Comparing (7.26) and (7.27), one can conclude that for $\tau > \tau_0^* > \tau_0$ the function F_1 is bounded from below $F_1 \geqslant F_1^* > 0$. Integrating (7.27) then yields

$$V^* \leqslant V(\tau_0) - \int_{\tau_0^*}^{\infty}\frac{F_1^*}{\tau}d\tau.$$

Since the integral is divergent, the assumption $V^* > 0$ is erroneous and therefore $V^* = 0$. Thus, the equilibrium is asymptotically stable. □

Remark 1. Due to the non-zero function F_0 in (7.27) there are solutions that exhibit exponential convergence (along some fixed directions) to the origin.

Let us examine the asymptotic behavior of solutions near the fixed points in greater detail.

a) Unstable equilibrium. Let $\rho = 1 + X$, $\psi + \varepsilon\varphi = \pi + Y$, $\varphi = Z$; the solution of the linearized system is

$$X = \frac{C_1}{\sqrt{\tau}}, \quad Y = \frac{C_2}{\sqrt{\tau}}, \quad Z = \left(C_3 - \sqrt{\frac{\pi}{\varepsilon}} C_2 \operatorname{erf}\left(\sqrt{\varepsilon\tau}\right)\right) e^{\varepsilon\tau}.$$

b) Stable equilibrium. Let $\rho = 1 + X$, $\psi + \varepsilon\varphi = \varepsilon\pi + Y$, $\varphi = \pi + Z$; the solution of the linearized system reads

$$X = \frac{C_1}{\sqrt{\tau}}, \quad Y = \frac{C_2}{\sqrt{\tau}}, \quad Z = \left(C_3 + C_2\sqrt{\frac{\pi}{\varepsilon}} \operatorname{erfi}\left(\sqrt{\varepsilon\tau}\right)\right) e^{-\varepsilon\varkappa\tau},$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt, \quad \operatorname{erfi}(z) = -i\operatorname{erf}(iz) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt,$$

and C_1, C_2, C_3 are constants due to integration. One can verify that the following asymptotic representation holds true $\sqrt{\pi} e^{-z} \operatorname{erfi}(\sqrt{z}) = \frac{1}{\sqrt{z}}\Big(1 + \frac{1}{2z} + \frac{3}{4z^2} + o(z^{-2})\Big)$.

Upon inspection of the expression for $\frac{d\rho}{d\tau}$ we conclude that

$$\rho(\tau) \leqslant 1 + (\rho_0 - 1)\sqrt{\frac{\tau_0}{\tau}}, \tag{7.28}$$

where $\rho_0 = \rho(\tau_0)$. To prove this, note that for $0 \leqslant \varepsilon < 1$ one gets $\cos\varphi\cos\psi - \varepsilon\sin\varphi\sin\psi \leqslant 1$ and therefore $\frac{d\rho}{d\tau} \leqslant -\frac{1}{2\tau}(\rho - 1)$. Integrating this inequality gives (7.28).

The physical meaning of (7.28) is that the most rapid growth of the body's kinetic energy is occurring in the straight down dive of the body.

Similarly it can be proved that

$$\rho(\tau) \geqslant (1 + \rho_0)\sqrt{\frac{\tau_0}{\tau}} - 1,$$

which means that the kinetic energy never decreases faster than in the upward straight drift.

It is shown in [58] that up to linear terms in χ almost all solutions tend to the second-type solutions of the form (7.22.2).

7.3. Qualitative Characteristics of Motion

An intriguing fact about the partial solution (7.22.2) is that the region where sufficient conditions for stability (with respect to part of the variables) are satisfied can be explicitly found. Consider the function

$$h = \frac{1}{2}w^2 + \varepsilon\cos\varphi, \tag{7.29}$$

By virtue of (7.19) we have

$$\frac{dh}{d\bar{t}} = -\varepsilon v w^2, \quad \frac{dv}{d\bar{t}} = \Big(\varepsilon + \frac{1}{2\varepsilon}\Big)w^2 - \frac{1}{\varepsilon}h. \tag{7.30}$$

These equations clearly show that if the inequalities $v > 0$ and $h < 0$ are satisfied at initial time $\bar{t} = 0$, then they hold true for all time because, in view of (7.30), $\frac{dv}{d\bar{t}}(\bar{t}) > 0$ and $\frac{dh}{d\bar{t}} < 0\ \forall\bar{t}$. Thus the following proposition is established.

Proposition 1. *Suppose that at $\overline{t} = 0$ the initial conditions lie in the domain defined by $v > 0$, $h < 0$; then for all $\overline{t} \geqslant 0$ the trajectory resides in this domain.*

Using the method from the proof of Theorem 1, which involves analysis of some divergent integral, one can show that for almost all initial conditions from the domain just mentioned

$$\varphi(\overline{t}) \xrightarrow[t\to+\infty]{} \pi.$$

This means that the center of mass tends to its lowermost possible position.

Now we will show that for $v > 0$ trajectories leave in finite time the domain of bigger values of h.

Proposition 2. *Let $t = 0$: $v = v_0$, $h = h_0 > \varepsilon + \frac{1}{2\varepsilon}$; then there exists an instant of time $t_* \leqslant v_0^{-1}\left(h_0 - \varepsilon - \frac{1}{2\varepsilon}\right)$ such that $h(t_*) \leqslant \varepsilon + \frac{1}{2\varepsilon}$.*

Proof. It follows from (7.19) that $\frac{dv}{dt}$ depends only on w, φ; the domain defined by the condition $\frac{dv}{dt} \leqslant 0$ encloses the origin of coordinates and the level lines of the function $h = \mathrm{const}$ are tangent to the boundary of this domain when $h = 0$ and $h = \varepsilon + \frac{1}{2\varepsilon}$ (Fig. 17).

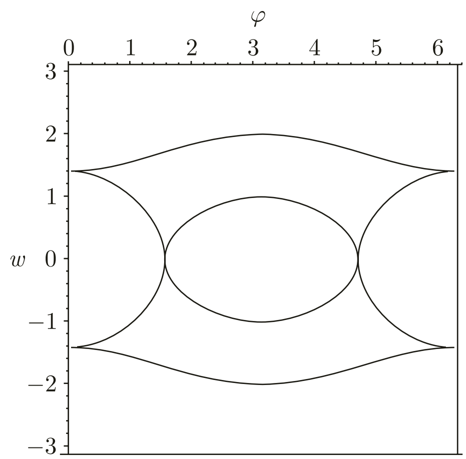


Fig. 17.

The figure indicates that in the domain $h > \varepsilon + \frac{1}{2\varepsilon}$ the inequality $\frac{dv}{dt} > 0$ is satisfied and therefore, if at initial time $v_0 > 0$, then by virtue of (7.30) the function $h(\overline{t})$ decreases monotonically; moreover, at each point of the domain $\frac{dh}{dt} \leqslant -v_0$. Hence, at some moment of time $t_* \leqslant v_0^{-1}\left(h_0 - \varepsilon - \frac{1}{2\varepsilon}\right)$ the value of h is less than $\varepsilon + \frac{1}{2\varepsilon}$.

Note that the equations of motion (7.19) remain unaltered under the change of variables

$$\overline{t} \to -\overline{t}, \quad v \to -v, \quad w \to -w, \quad \varphi \to \varphi. \tag{7.31}$$

As a consequence, we immediately conclude that the solution (7.22.2) (in which t is replaced by $-t$) is also stable for $t \to -\infty$. This conclusion is far from being intuitive because the center of mass is now behind the plate's midpoint; even it might seem to be quite natural instead to expect the solution (7.22.1) to be stable as $t \to -\infty$ because there the center of mass is ahead of the midpoint. Thus the nonholonomic plate subject to gravity resembles in a way a pendulum whose "pivot" is at the midpoint. In addition to this, the invariance of the equations under the change (7.31) implies that the region where the sufficient conditions for stability of (7.22.2) (as $\overline{t} \to -\infty$) are satisfied is defined by (see Proposition 1)

$$h < 0, \quad v < 0.$$

7.4. Computer Analysis

Simulations can help to extend the results obtained above. First let us examine the stability region described by Proposition 1. Let us denote the initial region (as $t \to 0$) defined by $v > 0$ and $h < 0$ as S_0 and see how this region S_t, $t < 0$, evolves with time. It is clear that the solutions emanating from the points of S_t will inevitably get into S_0 and therefore converge (as $t \to +\infty$) to the stable solution (7.22.2).

Schematics of part of the boundary of S_t for various values of ε and sufficiently small t are given in Figs. 18–20. Numerical evidence shows that the stability region is essentially larger (Figs. 18–20) compared to what has been established analytically (Proposition 1). Besides, basing on the results of simulations, we were led to the following hypothesis.

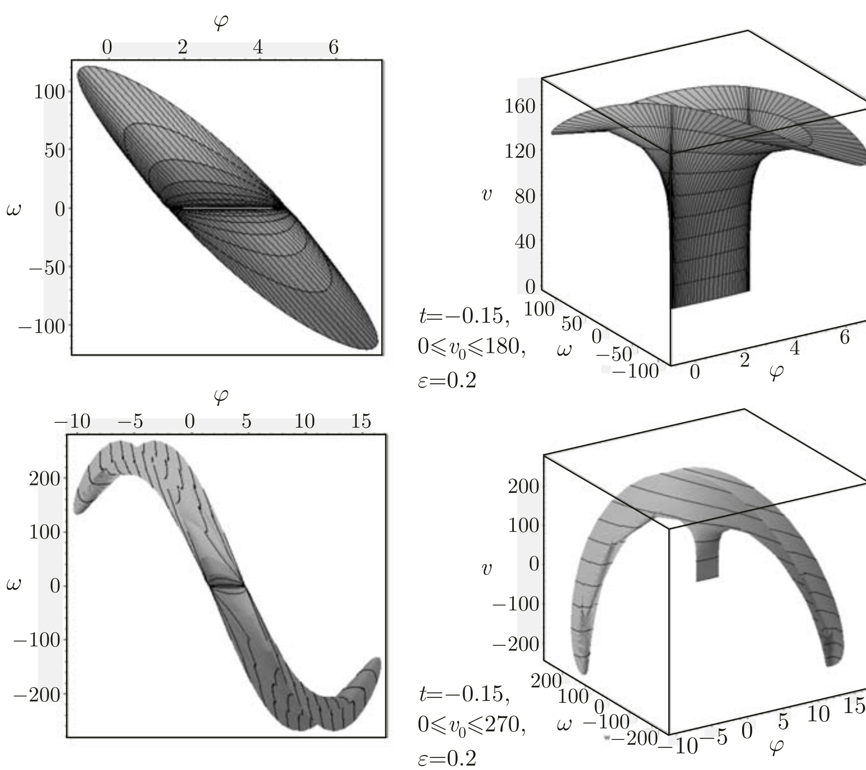


Fig. 18. Evolution of the boundary of the domain $h < 0$ (i.e. the surface $h = 0$) as time varies from 0 to $t < 0$. The two upper diagrams illustrate part of the surface between the planes $v = 0$ and $v = 180$, and for the lower diagrams these planes are $v = 0$ and $v = 270$. (For better visualization the points whose φ coordinates differ by an integral multiple of 2Π are not identified.)

Conjecture. *Almost all solutions of* (7.19) *tend to the solution* (7.22.2) *as* $\overline{t} \to +\infty$.

One interesting feature of the regions S_t should be mentioned. Suppose that $v < 0$ and $|v|$ is sufficiently large; then for small $|t|$ and $t < 0$ the boundary of S_t approaches arbitrarily close to the plane $w = 0$ (Figs. 18–20).

Thus summarizing the above stated one can conclude that for almost all initial conditions the nonholonomic plate tends to the vertical straight-dive motion ($t \to +\infty$) with its plane oriented vertically and the center of mass ahead the midpoint. Here the following question arises: how many revolutions does the plate make depending on the initial conditions at $t = 0$?

To explore this question numerically we put $t = 0$ and fix a plane $v = v_0 = \mathrm{const}$ in the space of variables v, w and φ. We start the solution from each point in this plane and then count the number of revolutions. Thus, depending on this number, the plane is divided into sets $D^{(n)}$, $n = 0, 1, \ldots$.

For better visualization points with equal numbers are colored the same (Fig. 21). It is clearly seen from the figure that the stability region given by Proposition 1 lies inside the union $D^{(0)} \cup D^{(1)}$, meaning that the body makes one revolution at best.

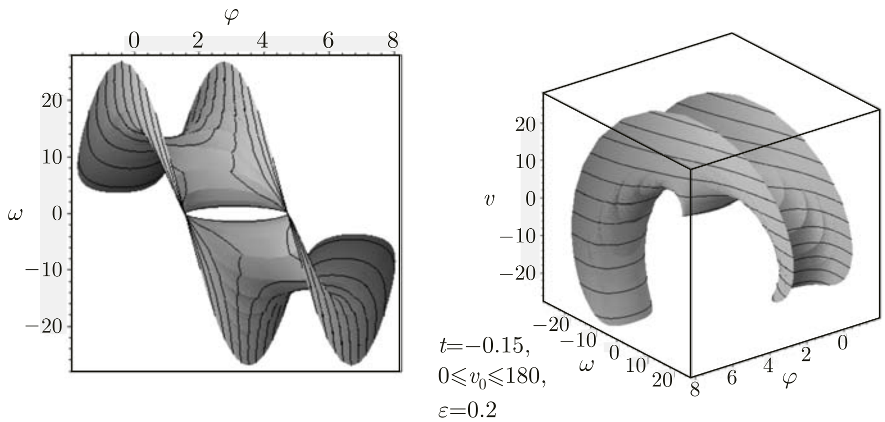


Fig. 19. A typical view of the boundary of the region S_t for sufficiently small ε.

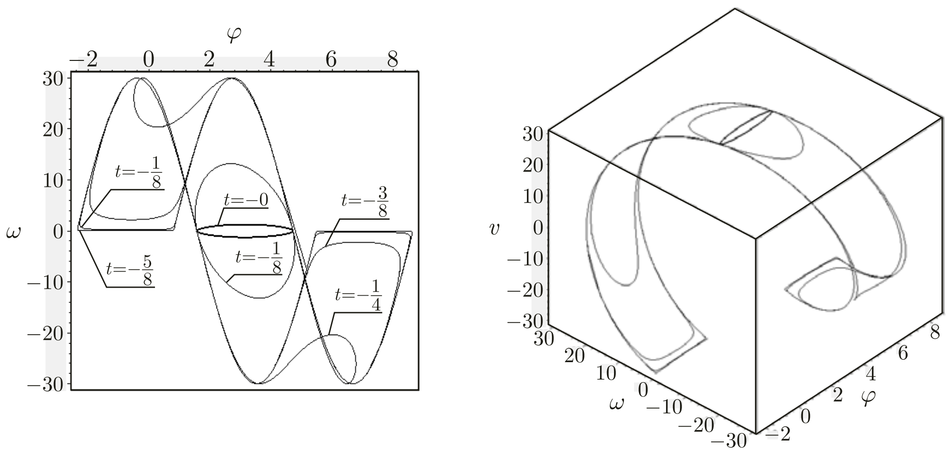


Fig. 20. Intersection of the region S_0 with the surface $h = 0$, $v = 30$ for $t = -\frac{1}{8}, -\frac{1}{4}, -\frac{3}{8}, -\frac{1}{2}, -\frac{5}{8}$ $(\varepsilon = 0.8)$.

A surprising result of simulations is that in some areas of the phase space the sets $D^{(n)}$ become practically inseparable (Fig. 21). Sometimes it was impossible to detect any regular boundaries between them even after "zooming up" to the scale comparable with the accuracy of computations.

In this connection, two questions arise that can hardly be resolved numerically.

1. Is it true that the partition $D^{(n)}$ of the initial-condition plane is in some sense of fractal type? (The sets $D^{(n)}$ exhibit intricate alternating behavior no matter how dense the grid of points on the plane is.)

2. Are there solutions with infinitely many revolutions and solutions for which the plate does not experience secular drift along the vertical? (According to the hypothesis stated above the union of such solutions is a null set at best.)

As for a plate moving in fluid, there is a variety of models of the Chaplygin sleigh on an inclined plane with friction taken into account. Let us consider two models. The first one preserves the nonholonomic constraint and adds dissipation due to friction of the supporting legs (not blade) on the plane. The motion of a sleigh on an inclined plane with viscous friction is considered in [42] where the stationary solutions are found and their linear stability investigated. In the context of the second model the slippery motion of the body is possible (a lateral displacement is allowed) and the constraint is replaced with the Coulomb friction with a sufficiently large coefficient. The sleigh on a plane with Coulomb friction was discussed in [59]. In particular, it was shown that the study of a dynamically balanced Chaplygin sleigh can be reduced to the study of a two-dimensional point mapping. The mapping can be explored only numerically because its structure is essentially complicated. Depending on a single essential parameter $\lambda \geq 0$, which determines the magnitude of the friction force, the body either converges to a steady state

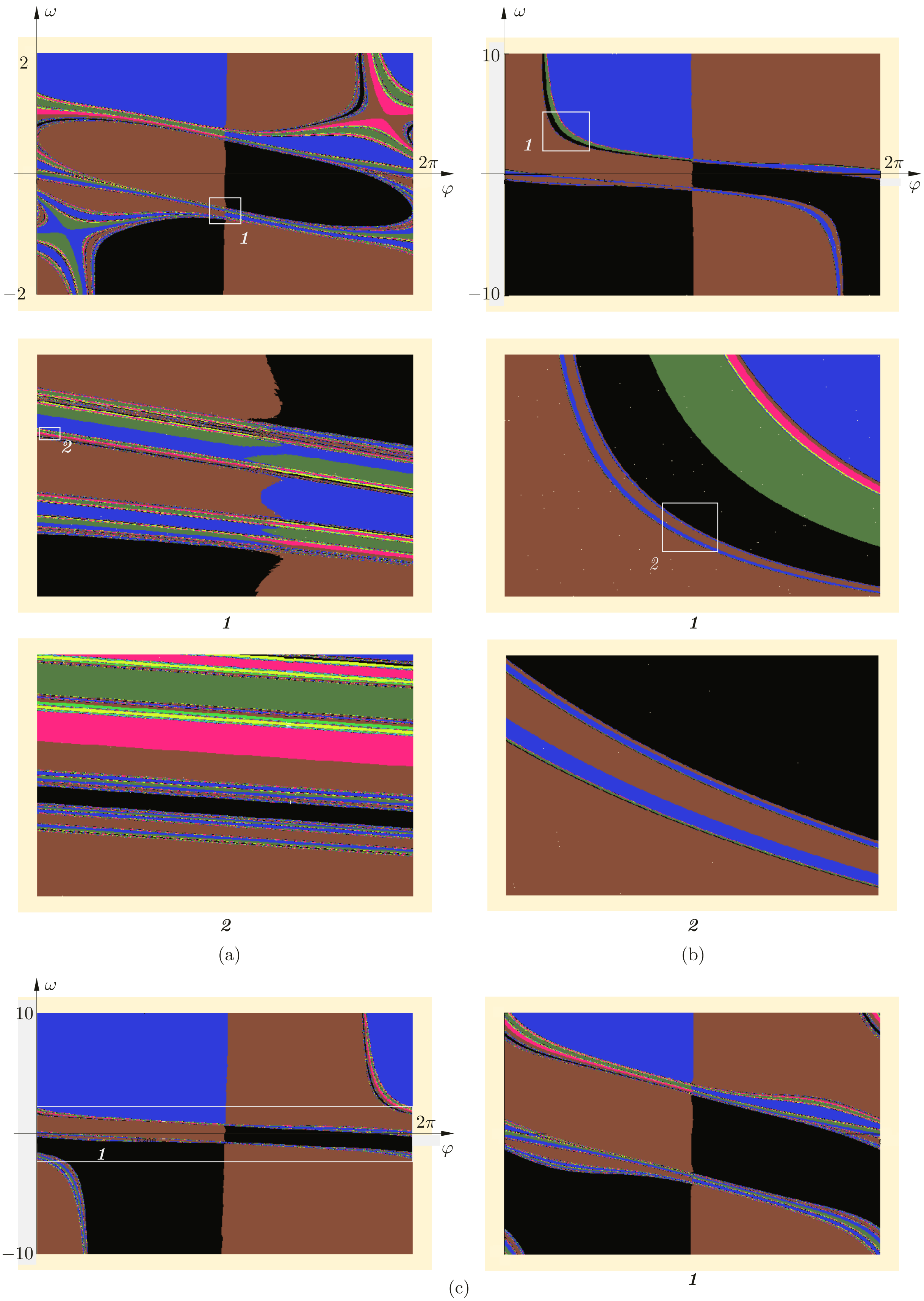


Fig. 21. A typical view of the partition $D^{(n)}$ on a plane of initial conditions: (a) $v_0 = 0$, $a = 1$, $\mu = 1$, $m = 0.5$, $I = 10$; (b) $v_0 = 0$, $a = 0.5$, $\mu = 1$, $m = 1$, $I = 3$; (c) $v_0 = 5$, $a = 1$, $\mu = 1$, $m = 0.5$, $I = 10$.

regime in which the center of mass does not (in average) drift down the plane($\lambda > 2$) or the center of mass is already drifting down the plane ($\pi/2 < \lambda < 2$). In the latter case the trajectory is composed of alternating identical curved segments of motion with and without slipping. Unfortunately, more general statements of this problem (say, the center of mass is shifted or the law of friction is not standard) are considerably complicated and have not been studied yet.

In conclusion, we would like to mention the problem of motion of a dynamically asymmetric balanced ball that rolls without slipping on an inclined plane was studied in [60]. Unlike the Chaplygin sleigh, this system possesses an invariant measure and, in some sense, its asymptotic behavior is more trivial.

8. FALL OF A PLATE IN A VISCOUS LIQUID UNDER THE INFLUENCE OF CIRCULATION

In the considered problem of a falling motion of a rigid body in fluid the surrounding fluid was assumed to be inviscid and the circulation around the body was chosen to be zero. To describe more adequately a real falling motion, one should take into consideration the dissipative effects due to the fluid's viscosity and, following the fundamental ideas of Joukowski and Chaplygin [21, 61, 62], one cannot neglect the circulatory motion around the body, which results in a lifting force.

The horizontal shift of a plate tossed in the air was first noticed by J.C. Maxwell [63] and L.P. Mouillard [64]. Consider a prolate rectangle of dense paper; suppose that at initial time the rectangle is rotating about its long axis, then it will preserve the direction of rotation and drift downwards along an inclined straight line; if the rectangle moves away from an observer then the observer sees its upper edge travel down. A rigorous mathematical analysis of this problem was undertaken by N.E. Joukowski [61, 62]. In the first of these papers within the framework of the theory of vortical flow around the plate neighborhood he considered a simplified model of interaction of a rigid body with the ambient medium. The interaction was modeled as the lifting force applied to the body (the angle of attack was assumed to be constant). Under the assumption that the circulation along any curve enclosing the plate is the same and does not change with time, the center of mass was shown to move in a horizontally oriented cycloid with zero average vertical displacement (just like the balanced Chaplygin sleigh [42]). The study of the second paper incorporates not only the lift but also the drag force. These forces were assumed to be quadratic functions of the velocity of the center of mass. The coefficients in these functions, which depend on the angle of attack, were replaced with their constant average values. Within this model, the body can travel downwards along an inclined line. In [61, 62] the angular velocity of the plate is constant. However, experiments show that in reality the angular velocity rapidly converges to a certain limit value which does not depend on the initial conditions and is solely determined by the physical parameters of the problem.

Plane-parallel motion of an arbitrarily shaped body in an ideal fluid in the presence of circulation around the body was first studied by Chaplygin [65] and Lamb [66]. In [65], S.A. Chaplygin derived the equations of motion, found and explored a case of integrability, and studied partial solutions. A modern analysis of these equations, the proof of non-integrability, and bibliography can be found in [67]. In contrast to the circulation-free model discussed above, a body with non-zero circulation will always remain in a horizontal strip [30], meaning that its average vertical displacement is zero. As already mentioned, this unrealistic result already follows from the simplest model due to N.E. Joukowski [61].

If dissipation is taken into consideration (introduced, for example, with a Rayleigh function), then the body's average vertical displacement is different from zero. In this model, there is a wide variety of possible regimes of motion as described by J. Maxwell. From different points of view, these regimes are discussed in [36, 68–73], where results of numerical and lab experiments on falling motion of a circular disk are reported. Papers [74, 75] discuss similar regimes for the fall of a round disk.

In the generic situation, there are two uniform motions of the disk: 1) the velocity of the center of mass is orthogonal to the disk's plane and 2) the velocity lies in this plane. The latter is always unstable. The first motion can cease being stable via the Hopf bifurcation resulting in occurrence of oscillatory regimes. Two such regimes are distinguished, which are related to each other by a heteroclinic bifurcation. These regimes are recognized to be one of the most beautiful hydrodynamic phenomena. In the first regime the body flutters in air and oscillate broadside on about now unstable uniform fall. On average, the body does not drift in the horizontal direction. Tumblings around the uniform fall are typical for the second regime. The horizontal average displacement (according to Mouillard's scenario) can be different from zero. Available analytical and numerical results on transition between these regimes are collected in the recent paper [68] (numerical results reported in the paper are obtained both from numerical analysis of phenomenological models and straightforward solution of the Navier—Stokes equation).

In [76] a chaotic regime arising in the problem of falling motion of a body in a viscous fluid is reported. A set that resembles a strange attractor occurs in the phase space. However, as justly mentioned in [77], the authors [76] neglected inertial properties of fluid (non-constant added inertia tensor). It seems to be interesting to examine such complicated regimes but with the added-mass effect taken into account.

In conclusion, we would like to mention the paper [78], which is modestly cited. The model considered in the paper incorporates viscous friction and allows for non-zero average vertical and horizontal displacements. Using the small parameter method, a stable periodic solution of the equations of motion is found. The solution is asymptotically stable with respect to part of variables. An integrable case reported in [78] can be studied in detail for large t. In this case the equations do not admit self-oscillations and the plate tends to a uniform straight line fall (for large t) with asymptotically constant velocity.

9. ACKNOWLEDGMENTS

The authors are thankful to K.G. Tronin for carrying out the computer calculations. This work was carried out under support of RFBR (project 05-01-01058), ERG “Regular and Chaotic Hydrodynamics” (RFBR Project 07-01-92210), the State Program for Support of Leading Scientific Schools NSh 1312.2006.1. Borisov and Mamaev also acknowledge the support from INTAS (project 04-80-7297).

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