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Invariant Planes, Indices of Inertia, and Degrees of Stability of Linear Dynamic Equations

V. V. Kozlov [a]

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Abstract—Spectral properties of linear dynamic equations linearized at equilibrium points are analyzed. The analysis involves a search for invariant planes that are uniquely projected onto the configuration plane. In turn, the latter problem reduces to the solution of a quadratic matrix equation of special form. Under certain conditions, the existence of two different solutions is proved by the contraction mapping method. An estimate for the degree of stability is obtained in terms of the index of inertia of potential energy.

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1. INTRODUCTION. THE MAIN RESULT

Consider a mechanical system with n degrees of freedom subjected to potential, dissipative, and gyroscopic forces. The equations of motion linearized at an equilibrium point are expressed as follows:

$$M\ddot{x} + \Gamma\dot{x} + D\dot{x} + Px = 0, \qquad x \in \mathbb{R}^n. \tag{1.1}$$

Here, M is a positive definite matrix that defines the inertial properties of the system, so $(M\dot{x}, \dot{x})/2$ is the kinetic energy of the system. The skew-symmetric matrix Γ ($\Gamma^{\mathrm{T}} = -\Gamma$) is the matrix of gyroscopic forces. The symmetric $n \times n$ matrix $D \geq 0$ (D is positive semidefinite) is involved in the definition of the Rayleigh dissipation function $\Phi = -(D\dot{x}, \dot{x})/2$ ($\Phi \leq 0$). Finally, P is the matrix of potential forces ($P^{\mathrm{T}} = P$), which defines the potential energy of the system, $(Px, x)/2$.

We will assume that $|P| \neq 0$. Thus, $x = 0$ is a unique equilibrium point of the linear system (1.1).

Applying nondegenerate linear transformations of the coordinates x, one can reduce the matrix M to the identity matrix. In what follows, we will assume that $M = E$. Set $K = D + \Gamma$ and suppose that this matrix is nondegenerate. If $D = 0$, then we will assume that n is even. On the other hand, if the matrix D is positive definite, then $|K| = |D + \Gamma| > 0$ for any skew-symmetric matrix Γ [1].

The total energy of the mechanical system is given by the quadratic form

$$H = \frac{(\dot{x}, \dot{x})}{2} + \frac{(Px, x)}{2}. \tag{1.2}$$

It is clear that $\dot{H} = 2\Phi \leq 0$. Let p be the *index of inertia* of the quadratic form (1.2); this is the *Poincaré degree of instability of system* (1.1). Obviously, p coincides with the index of inertia of the potential energy.

The *degree of instability* u of system (1.1) is defined as the number of roots of the characteristic equation

$$|\lambda^2 E + \lambda K + P| = 0 \tag{1.3}$$

that lie in the right complex half-plane.

[a] Steklov Institute of Mathematics, Russian Academy of Sciences, ul. Gubkina 8, Moscow, 119991 Russia.
E-mail address: kozlov@pran.ru

If $K=0$, then $u=p$. In the general case, $u \equiv p \mod 2$ (Thomson's theorem). The number $s = 2n-u$ is called the *degree of stability* of the linear system (1.1). Since $|P| \neq 0$ by our assumption, equation (1.3) has no zero roots.

Let $\|\cdot\|$ be the operator norm of a matrix. Recall that

$$\|X\| = \max_{\|x\|=1} \|Xx\|.$$

In particular, $\|E\| = 1$ and $\|X^{\mathrm{T}}\| = \|X\|$. As a rule, these relations are not valid for other matrix norms. It is clear that the operator norm depends on the choice of norm in the space $\mathbb{R}^n = \{x\}$.

The main result of this paper is the following theorem.

Theorem 1. *If*

$$\|K^{-1}\| \cdot \|K^{-1}P\| < \frac{1}{4} \qquad \textit{and} \qquad \|K^{-1}\| \cdot \|PK^{-1}\| < \frac{1}{4}, \tag{1.4}$$

then

(1) *the phase space* $\mathbb{R}^{2n} = \{x, \dot{x}\}$ *of system* (1.1) *is the direct sum of invariant* n*-dimensional planes*

$$\Sigma = \{\dot{x} = Ax\} \qquad \textit{and} \qquad \Sigma' = \{\dot{x} = A'x\},$$

where A *and* A' *are nondegenerate* $n \times n$ *matrices such that*

$$\|A\| \le \frac{1 - \sqrt{1 - 4\|K^{-1}\| \cdot \|K^{-1}P\|}}{2\|K^{-1}\|}, \qquad A' = -K + B,$$

$$\|B\| \le \frac{1 - \sqrt{1 - 4\|K^{-1}\| \cdot \|PK^{-1}\|}}{2\|K^{-1}\|};$$

(2) *the restrictions of the total energy* (1.2) *to* Σ *and* Σ' *are quadratic forms* h *and* h', *the latter being positive definite*; *if* $D>0$ *or* $D=0$, *then* h *is a nondegenerate quadratic form with the index of inertia* p;

(3) *by virtue of the linear systems of differential equations*

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \tag{1.5}$$

and

$$\dot{x} = A'x, \qquad x \in \mathbb{R}^n, \tag{1.6}$$

the derivatives of the quadratic forms h *and* h' *are equal to*

$$-(DAx, Ax) \qquad \textit{and} \qquad -(DA'x, A'x),$$

respectively.

In the absence of dissipation (when $D=0$), Theorem 1 was established in [2]. In this case, the two conditions in (1.4) coincide. Indeed,

$$\|\Gamma^{-1}P\| = \|(\Gamma^{-1}P)^{\mathrm{T}}\| = \|P(-\Gamma)^{-1}\| = \|P\Gamma^{-1}\|.$$

It may happen that the quadratic form h is equivalent to the potential energy in the general case of $D \ge 0$ under conditions (1.4). Until now this fact has been proved only under the stronger condition

$$\|K^{-1}P\|^2\, \|P^{-1}\| < \frac{1}{4} \tag{1.7}$$

(see Section 2).

Inequalities (1.4) (just as (1.7)) certainly hold when the dissipative and gyroscopic forces dominate the potential forces. This statement can be given the following exact meaning. Let us replace the matrices K and P in equation (1.1) by μK and νP, respectively, where μ and ν are real parameters. Then, inequalities (1.4) and (1.7) turn out to be valid for sufficiently large values of the ratio $\mu^2/|\nu|$.

According to Theorem 1, the analysis of the linear system (1.1) of order $2n$ reduces to the analysis of two systems of linear differential equations, (1.5) and (1.6), of order n. In particular, the spectrum of system (1.1) is the union of the spectra of the linear systems (1.5) and (1.6). The equilibrium of system (1.6) is always stable. Moreover, in the case of complete dissipation (i.e., $D > 0$), all eigenvalues of the matrix A' lie in the left half-plane. Thus, the degrees of instability of systems (1.1) and (1.5) coincide.

Theorem 1 is proved in Section 2. Virtually all the arguments can be carried over word for word to the infinite-dimensional case, when the configuration space $\mathbb{R}^n = \{x\}$ is replaced by a Hilbert space and the matrices are replaced by operators with appropriate properties.

2. INVARIANT PLANES

Lemma 1. *The plane*

$$\Sigma = \{(x, \dot{x}) \in \mathbb{R}^{2n} :\ \dot{x} = Ax\} \tag{2.1}$$

is invariant under the phase flow of system (1.1) *if and only if*

$$(MA + K)A + P = 0. \tag{2.2}$$

Indeed, if Σ is invariant, then (according to (1.1))

$$MA\dot{x} = M\ddot{x} = -K\dot{x} - Px.$$

Applying again the relation $\dot{x} = Ax$, we obtain

$$(MA^2 + KA + P)x = 0$$

for any $x \in \mathbb{R}^n$. This implies the matrix equation (2.2). The converse assertion is proved along the same line.

Does the matrix equation (2.2) have solutions and, if yes, how many? This is a nontrivial problem: one can produce simple examples of quadratic matrix equations that have a continuum of solutions, as well as equations that do not admit any solution (even with complex elements).

First, suppose that $\|\cdot\|$ is an *arbitrary* matrix norm.

Lemma 2. *If*

$$\|K^{-1}M\| \cdot \|K^{-1}P\| < \frac{1}{4}, \tag{2.3}$$

then equation (2.2) *admits a unique solution that satisfies the condition*

$$\|A\| \le \frac{1 - \sqrt{1 - 4\|K^{-1}M\| \cdot \|K^{-1}P\|}}{2\|K^{-1}M\|}. \tag{2.4}$$

Proof. By the assumption, the matrix K is invertible. Therefore, equation (2.2) can be rewritten as follows:

$$A = F(A) = -K^{-1}(MA^2 + P).$$

Let S be the closed ball defined by inequality (2.4) in the space of $n \times n$ matrices. If $A \in S$, then

$$\begin{aligned}\|F(A)\| &\le \|K^{-1}M\| \cdot \|A\|^2 + \|K^{-1}P\| \\ &\le \frac{\left(1 - \sqrt{1 - 4\|K^{-1}M\| \cdot \|K^{-1}P\|}\,\right)^2 + 4\|K^{-1}M\| \cdot \|K^{-1}P\|}{4\|K^{-1}M\|} \\ &= \frac{1 - \sqrt{1 - 4\|K^{-1}M\| \cdot \|K^{-1}P\|}}{2\|K^{-1}M\|}.\end{aligned}$$

So, the nonlinear operator F maps S into itself.

Next, for any A and A' in the ball S, we have

$$\begin{aligned}\|F(A) - F(A')\| &= \left\|-K^{-1}M\bigl(A^2 - AA' + AA' - A'^2\bigr)\right\| \\ &\le \|-K^{-1}M\| \left(\|A\| + \|A'\|\right) \|A - A'\| \\ &\le \left(1 - \sqrt{1 - 4\|K^{-1}M\| \cdot \|K^{-1}P\|}\,\right)\|A - A'\|.\end{aligned}$$

Since the expression in parentheses is less than unity, F is a contraction operator. Hence, it has a unique fixed point in the complete metric space S, which was to be proved.

By analogy with ordinary quadratic equations, it is natural to assume that the matrix equation (2.2) admits another real solution. However (since the multiplication of matrices is not commutative), this analogy is inconsistent. Nevertheless, the following proposition is valid.

Lemma 3. *If*

$$\|MK^{-1}\| \cdot \|PK^{-1}\| < \frac{1}{4}, \tag{2.5}$$

then equation (2.2) *admits a solution*

$$A' = M^{-1}BM - M^{-1}K \tag{2.6}$$

with

$$\|B\| \le \frac{1 - \sqrt{1 - 4\|MK^{-1}\| \cdot \|PK^{-1}\|}}{2\|MK^{-1}\|}. \tag{2.7}$$

If $D = 0$, then one should set $B = A^{\mathrm{T}}$ in formula (2.6). Then (2.6) turns into one of Viète's formulas. Another of Viète's formulas reduces to the equality

$$A^{\mathrm{T}}MA' = -P. \tag{2.8}$$

Formulas (2.6) (with $B = A^{\mathrm{T}}$ and $K = \Gamma$) and (2.8) are invariant under the replacement of A by A'. This duality is of symmetric nature and was discussed in [2]. Let us also point out a factorization formula for a quadratic matrix pencil:

$$\lambda^2 M + \lambda\Gamma + P = \bigl(\lambda E + A'^{\mathrm{T}}\bigr)(\lambda M - MA).$$

Proof of Lemma 3. Let us make the substitution (2.6) in the original equation (2.2). Then B satisfies the following quadratic equation:

$$B(BM - K) + P = 0.$$

We rewrite this equation as

$$B = \Phi(B) = B^2MK^{-1} + PK^{-1}$$

and apply the scheme of the proof of Lemma 2.

Since A and A' satisfy equation (2.2) and $|P| \neq 0$, these matrices are nondegenerate. To complete the proof of assertion (1) of Theorem 1, it remains to show that the n-dimensional planes Σ and Σ' intersect only at zero. In turn, this is equivalent to the nondegeneracy of the matrix $A - A'$.

From now on, we assume that $\|\cdot\|$ is the operator norm and M is the identity matrix.

Lemma 4. *Under conditions* (2.3) *and* (2.5), *we have* $|A - A'| \neq 0$.

Proof. Let us apply the identity

$$K^{-1}(A - A') = K^{-1}A - K^{-1}B + E. \tag{2.9}$$

First, we estimate $\|K^{-1}A\|$ using inequality (2.4):

$$\|K^{-1}A\| \leq \|K^{-1}\| \cdot \|A\| \leq \frac{1 - \sqrt{1-z}}{2},$$

where $z = 4\|K^{-1}\| \cdot \|K^{-1}P\| > 0$. By condition (2.3), $z < 1$. Since

$$1 - \sqrt{1-z} < 1, \qquad 0 < z < 1,$$

we have

$$\|K^{-1}A\| < \frac{1}{2}.$$

Analogously (using estimate (2.7)) one can prove that

$$\|K^{-1}B\| < \frac{1}{2}. \tag{2.10}$$

Thus,

$$\|K^{-1}A - K^{-1}B\| < 1;$$

therefore, the matrix (2.9) is invertible, which was to be proved.

In particular, $\mathbb{R}^{2n}$ is the direct sum of the n-dimensional invariant subspaces

$$\Sigma = \{\dot{x} = Ax\} \qquad \text{and} \qquad \Sigma' = \{\dot{x} = A'x\},$$

which are uniquely projected onto the configuration space $\mathbb{R}^n = \{x\}$.

Let h and h' be the restrictions of the total energy (1.2) to Σ and Σ', respectively. It is clear that

$$h(x) = \frac{1}{2}\big(A^{\mathrm{T}}Ax, x\big) + \frac{1}{2}(Px, x), \tag{2.11}$$

$$h'(x) = \frac{1}{2}\big(A'^{\mathrm{T}}A'x, x\big) + \frac{1}{2}(Px, x). \tag{2.12}$$

Lemma 5. *By virtue of systems* (1.5) *and* (1.6), *the derivatives of the functions h and h' are equal to*

$$-(Ax, DAx) \qquad \textit{and} \qquad -(A'x, DA'x),$$

respectively.

Indeed, Σ and Σ' are invariant planes, and $\dot{H} = -(D\dot{x}, \dot{x})$.

Lemma 6. *If inequality* (2.5) *holds, then the quadratic form h' is positive definite.*

Corollary 1. *Under assumption* (2.5), *the equilibrium point $x = 0$ of the linear system is always stable, and if $D > 0$, it is asymptotically stable.*

Proof of Lemma 6. Let us make the linear substitution $A'x = z$ in (2.12). Then

$$h'(z) = \frac{1}{2}(z,z) + \frac{1}{2}\big(PA'^{-1}z, A'^{-1}z\big).$$

Let us estimate

$$\begin{aligned}\big\|\big(A'^{-1}\big)^{\mathrm{T}}PA'^{-1}\big\| &= \big\|\big(B^{\mathrm{T}} - K^{\mathrm{T}}\big)^{-1}P(B-K)^{-1}\big\| \\ &= \big\|\big(B^{\mathrm{T}}(K^{\mathrm{T}})^{-1} - E\big)^{-1}(K^{\mathrm{T}})^{-1}PK^{-1}(K^{-1}B - E)^{-1}\big\|.\end{aligned} \tag{2.13}$$

It is clear that

$$\begin{aligned}\big\|\big(B^{\mathrm{T}}(K^{\mathrm{T}})^{-1} - E\big)^{-1}\big\| &= \|E\| + \big\|B^{\mathrm{T}}(K^{\mathrm{T}})^{-1}\big\| + \big\|B^{\mathrm{T}}(K^{\mathrm{T}})^{-1}\big\|^2 + \dots \\ &\le 1 + \|K^{-1}B\| + \|K^{-1}B\|^2 + \dots < 2\end{aligned}$$

according to (2.10). The estimate

$$\big\|(K^{-1}B - E)^{-1}\big\| < 2$$

is proved in a similar way. Hence, the right-hand side of (2.13) is no greater than

$$4\big\|(K^{\mathrm{T}})^{-1}\big\| \cdot \|PK^{-1}\| = 4\|K^{-1}\| \cdot \|PK^{-1}\| < 1$$

by virtue of inequality (2.5). The lemma is proved.

If $\|A\|^2 < 1/\|P^{-1}\|$, then the index of inertia of the other quadratic form h is equal to p. This inequality (in view of estimate (2.4)) takes the form of condition (1.7).

Lemma 7. *Suppose that condition* (1.4) *holds and either $D > 0$ or $D = 0$. Then, the index of inertia of the quadratic form h is equal to p.*

Proof. If $D > 0$, then the degree of instability of the linear system (1.1) is equal to p [3]. Therefore (by Lemma 6), the degree of instability of the linear system (1.5) is also equal to p. It remains to apply the Ostrowski–Schneider theorem [4], which states that (in view of Lemma 5) the index of inertia of the form h coincides with the degree of instability of system (1.5).

For $D = 0$ the assertion of Lemma 7 was proved in the book [5, Ch. I].

It remains unclear whether Lemma 7 is valid in the general case $D \ge 0$ under condition (1.4).

3. ESTIMATE FOR THE DEGREE OF STABILITY OF SYSTEMS WITHOUT ENERGY DISSIPATION

Consider the conservative case, when there is no dissipation of energy ($D = 0$). Then, it is obvious that (1.1) are Hamiltonian equations and n is even. Let again p be the index of inertia of the potential energy.

Theorem 2. *If*

$$\|\Gamma^{-1}\| \cdot \|\Gamma^{-1}P\| < \frac{1}{4}, \tag{3.1}$$

then the degree of instability s of system (1.1) *is no less than*

$$\frac{3n}{2} + \frac{|n-2p|}{2}. \tag{3.2}$$

The graph of this function of p is shown in the figure. Let us discuss this result. If $p = 0$, then $s = 2n$ (according to (3.2)). This is a trivial corollary to the energy integral, which is naturally

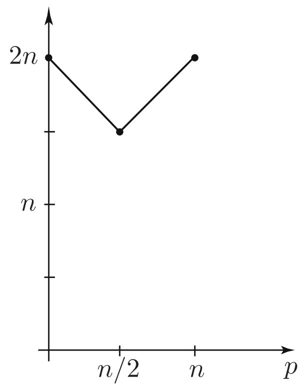

Figure.

valid without constraint (3.1). Similarly, if $p = n$ (the potential energy has a strict maximum at the equilibrium point), then again $s = 2n$. This is a meaningful result from the theory of gyroscopic stabilization, and condition (3.1) is essential here. Earlier, the gyroscopic stabilization condition was presented by S.V. Bolotin in the form

$$\|\Gamma^{-1}\| \cdot \left\|\sqrt{-P}\,\right\| < \frac{1}{2}, \tag{3.3}$$

where $\sqrt{-P}$ is a positive definite symmetric matrix whose square is equal to $-P$ (recall that $P < 0$ in the case under consideration). This result can be found in the book [5]. Note that condition (3.1) is weaker than (3.3). Indeed,

$$\|\Gamma^{-1}\| \cdot \|\Gamma^{-1}P\| \le \|\Gamma^{-1}\|^2 \, \|-P\| \le \|\Gamma^{-1}\|^2 \left\|\sqrt{-P}\,\right\|^2 < \frac{1}{4}$$

provided that condition (3.3) holds.

Next, if $p = 1$, then $s = 2n-1$. This is a simple corollary to the general fact that the stability is lost when one negative square arises in the diagonal form of the potential energy: two eigenvalues turn out to be real with opposite signs, while the other eigenvalues are pure imaginary without Jordan cells (see [6] for detail). This fact is valid irrespective of condition (3.1).

The case $p = n - 1$ is more interesting. Here again $s = 2n - 1$, but now inequality (3.1) is essential. The structure of the spectrum of the linear system (1.1) is the same as in the case of $p = 1$. Note that if $p = n-1$ (and n is even), then $u \ge 1$ (Thomson's theorem). Thus, for sufficiently high intensities of gyroscopic forces, one can minimize the degree of instability by making it equal to unity.

When $p \ge 2$ and $p \le n - 2$, we can only assert that

$$s \ge \frac{3n}{2} + \frac{|n - 2p|}{2}. \tag{3.4}$$

Moreover, this estimate is sharp: there are simple examples in which the equality in estimate (3.4) holds under condition (3.1).

Proof of Theorem 2. First, conditions (2.3) and (2.5) coincide provided that $K = \Gamma$. Second, the quadratic form h' is positive definite by Lemma 6, and the index of inertia of the nondegenerate form h is equal to p by Lemma 7. If $D = 0$, then the linear systems (1.5) and (1.6) admit the first integrals h and h', respectively. Hence, all eigenvalues of the linear system (1.6) are pure imaginary without Jordan cells. According to [6], the number of pairs of pure imaginary eigenvalues of the linear system (1.5) is no less than $|(n-p)-p|/2$. Let u' be the degree of instability of system (1.5).

Since system (1.5) admits a nondegenerate quadratic integral, its spectrum is symmetric with respect to the imaginary axis [7]. Therefore, $n - 2u' \geq |n - 2p|$. Thus,

$$s = n + (n - u') \geq \frac{3n}{2} + \frac{|n-2p|}{2},$$

which was to be proved.

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